KEY

TO ROBINSON'S

UNIVERSITY ALGEBRA;

CONTAINING, ALSO,

A SHORT TREATISE ON THE INDETERMINATE
AND DIOPHANTINE ANALYSIS.

AND

SOME MISCELLANEOUS EXAMPLES.

Designed for Teachers and Students.

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PREFACE.

A KEY, to a mathematical work, is very proper in its place; but to be constantly at hand, and consulted too often, might prove injurious: we must not, however, confound the improper use of a thing with the thing itself. Those who condemn keys, in general terms, should condemn teachers also; for a key is neither more nor less than a teacher, in another shape.

The self-taught are generally sound and vigorous; but if they disregard the works and teachings of others, they will be found to be wanting in that certain symmetry and polish of mind, so characteristic of educated men.

So it is with an algebraist; he may go through his text-books, solve every problem, independent of all external aid, and if he does not compare his work with the works of others, he cannot know whether he is skillful or otherwise; for it is only by comparison that we measure excellence. No solution of a problem, or of an equation, should be called good, if better can be found; hence it is important that more than one standard of attainment should be before the pupil; and those who really become eminent, in any science, are those whose talents and dispositions enable them to gather knowledge from every possible source.

PREFACE TO REVISED EDITION.

In 1857 the author and publisher thought it advisable to enlarge the University Algebra by some twenty-four pages, and some more than twenty additional examples.

Solutions of these are inserted subsequent to page 76. Also, solutions of several other problems are inserted, which were omitted in the former editions of this work.

KEY

TO

ROBINSON'S ALGEBRA.

SECTION II.

CHAPTER I.

EQUATIONS.

None of the questions in this chapter require the aid of a key, until we come to the 15th, page 65.

(15.) $$\left(\frac{4x-4a}{3}-a\right)\frac{4}{3}=\frac{16x-16a}{9}-\frac{4a}{3}=$$ his stock at the commencement of the third year, before his expenses are taken out.

Hence, $$\left(\frac{16x-16a}{9}-\frac{4a}{3}-a\right)\frac{4}{3}=2x$$

Reduced gives $x=14800$, *Ans.*

(16.) Put $a=99$, $x=$time past. Then $a-x=$ time to come, and per question,

$$\frac{2x}{3}=\frac{4a-4x}{5}\ \ldots\ldots\ldots\ x=54.$$

(17.) Let $x=$ the whole composition.

Then per question,

$$\frac{2x}{3}+10=\text{nitre.}$$

$$\frac{x}{6}-4\tfrac{1}{2}=\text{sulphur.}$$

$$\frac{2x}{21}+\frac{10}{7}-2=\text{charcoal.}$$

By addition, $\frac{2x}{3}+\frac{x}{6}+\frac{2x}{21}+3\frac{1}{2}+\frac{10}{7}=x$

Multiply by 6, and drop $5x$ from both sides, and we have

$$\frac{4x}{7}+21+\frac{60}{7}=x$$

or, $4x+21\cdot7+60=7x$ . . . . $x=69$.

(18.) Put $a=183$; $x=$what the 1st received; then $a-x=$2d received.

Per question, $\frac{4x}{7}=\frac{3a-3x}{10}$ . . . . $x=63$.

(19.) Put $a=68$, $x=$ the greater part, and $a-x=$ the less. $84-x=3(40-a+x)$ . . . . . . $x=42$.

(20.) The distance from A to B put $=2x$.
The distance from C to D " $=3x$.
Then, 3 times the distance from B to C must be

$$\frac{x}{2}+\frac{3x}{2} \text{ or the distance is, } \frac{x}{6}+\frac{x}{2}$$

Hence the whole distance is, $5x+\frac{x}{6}+\frac{x}{2}=34$.

(21.) Let $x=$the flock.

The first party left him $\frac{2x}{3}-6$.

The second left $\frac{x}{3}-3-10=2$.

(23) Observe that for every vessel he broke he lost 12 cents: 3 cents fee and 9 cents forfeiture.

$$300-12x=240 \ldots\ldots\ldots\ldots x=5.$$

(24.) Had he not been idle he would have been entitled to ab cents. But he was idle x days at a loss of $(b+c)$ cents. Hence, $ab-(b+c)x=d$.

$$x=\frac{ab-d}{b+c}$$

(25.) Put $5x=$ less part

Then $a-5x=$ the greater part.

Per question, $a-7x=20x-\frac{3}{7}(a-5x)$

$$7a-49x=140x-3a+18x$$

or, $204x=10a=10\cdot204$

or, $x=10$

Therefore, $5x=50=$ the less part.

(26.) Let $8x=$the price of the horse.

Then $a-8x=$ chaise. $a=341$.

Per question, $2a-16x-3x=24x-\frac{5}{7}(a-8x)$

or, $2a=43x-\frac{5}{7}(a-8x)$

$$14a=301x-5a+40x$$

$19a=341x$ or, $x=19$

$8x=152$ *Ans.*

(28.) Let $5x=$ his money.

After he first lost and won 4 *s*., he had $4x+4$

He again lost and won, and then had $3x+3+3$.

$\frac{5}{6}$ of this must equal 20, or, $3x+6=24$.

$x=6$

$5x=30$ *Ans.*

(29.) Let $3x=$ the income.

Then $2x=$ the family support.

$$x-\frac{2x}{3}=\frac{x}{3}=70. \quad \text{Hence,} \ldots\ldots 3x=70\cdot 9.$$

30, 31, 32, and 33 require no explanation.

(35.) Last year the rent was x dollars.

This year it is $x+\frac{8x}{100}=1890$

(36.) is the (35) in general terms.

(37.) Let $7x=$ the income.

Then $x=$ A's annual debt.

$\frac{7x}{5}=$ what B saves.

$$\frac{7x}{5}-x=16 \qquad \text{or} \qquad x=40.$$

$$7x=280, \textit{ Ans.}$$

In general terms,

$$\frac{7x}{5}-x=\frac{a}{2} \qquad x=\frac{5a}{4}$$

$$7x=\tfrac{1}{4}(35a).$$

(38.) $$\frac{x}{3}+\frac{x}{4}+\frac{2x}{7}=a$$

(39.) Let $10x=$ the income.

Then $2x+100=$the sum spent.

$5x+\ 35=$ " sum left.

$7x+135=10x$ the whole

or $45=x \ldots\ldots\ldots 450$ *Ans*

(40.) Let $21x=$ the income.

Then $3x+a=$ the sum spent.

$7x+b=$ " sum left.

$10x+a+b=21x=$ the whole.

$$x=\frac{(a+b)}{11}.$$

(41.) $2x+4 \; : \; 3x+4 \; : : \; 5 \; : \; 7$

(42.) Let $x^2-7=$ the number.

Then, per conditions, $x-1=\sqrt{x^2-7}$

$$x^2-2x+1=x^2-7$$

or, $x=4$ and $x^2-7=9$.

(43.) A's rate of travel is $\frac{7}{5}$ miles per hour.

B's rate of travel is $\frac{5}{3}$ miles per hour.

A is in advance when B sets out, $\frac{56}{5}$ miles.

Let $x=$ the hours after B starts.

Then, $\frac{5x}{3}=\frac{7x}{5}+\frac{56}{5}$ Reduced gives ..... $x=42$.

CHAPTER II.

EQUATIONS IN TWO UNKNOWN QUANTITIES.

(6.) Add the two equations together, representing $(x+y)$ by s, and we have $\frac{1}{3}s+3s=50$ or ...... $s=5 \cdot 3$.

$$\begin{array}{lr} \text{But} & x+9y=21\cdot 3 \\ \text{Subtract} & x+\;y=\;5\cdot 3 \\ \hline & 8y=16\cdot 3 \end{array}$$

or, ... $y=6$.

(7.) By adding the two equations we have

$$5s=50$$

$$\begin{array}{lr} \text{or,} & x+y=10 \\ \text{but} & 4x+y=34 \\ \hline \text{Hence,} & 3x \quad =24 \end{array}$$

or, ... $x=8$.

(9) and (10) are resolved same as (6) and (7).

(11.) From the first equation we have

$$y=2x-80.$$

Transpose -8 in the second equation and we have

$$\frac{x+y}{5}+\frac{x}{3}=\frac{2y-x}{4}+35.$$

Multiply by 60 and we have

$$12x+12y+20x=30y-15x+35\cdot60$$

or $$47x=18y+35\cdot60$$

Substituting the value of $18y$, we have

$$47x=36x-18\cdot80+35\cdot60$$

or $$11x=-240\cdot6+350\cdot6=110\cdot6$$

Hence, ... $x=60.$

(14.) Bringing unknown terms to the first members of the equation and we have

$$\frac{4}{x}-\frac{4}{y}=-1 \qquad \frac{4}{y}-\frac{2}{x}=\frac{3}{2}$$

By addition, $\frac{2}{x}=\frac{1}{2}$ or . . . . $x=4.$

(15.) Put $a=50$.

Then, $x+3a : y-a :: 3 : 2$

And $x-a : y+2a :: 5 : 9$

$$2x+6a=3y-3a \qquad (1)$$

$$9x-9a=5y+10a \qquad (2)$$

Multiply (1) by 5, and (2) by 3; then,

$$10x+30a=15y-15a \qquad (3)$$

$$27x-27a=15y+30a \qquad (4)$$

Subtract (3) from (4) and

$$17x-57a=45a$$

$$17x=102a$$

$$x=6a=6\cdot50=300.$$

(16.) Divide the numerator of the second member of the first equation by its denominator, and we have

$$3x+6y+1=3x+6y+1+\frac{-11x-14y+127y}{2x-4y+3}$$

Hence, $11x+14y=127$ (1)

Multiply the second equation by $(3y-4)$ and we shall have

$$9xy-12x=\frac{(151-16x)(3y-4)}{4y-1}+9xy-110$$

or, $$110-12x=\frac{(151-16x)(3y-4)}{4y-1}$$

$$440y-48xy-110+12x=453y-48xy-604+64x$$

$$0=13y+52x-494$$

or, $$4x+y=38 \qquad (2)$$

Add (1) and (2), and we have

$$15(x+y)=165$$

or, $$x+y=11 \qquad (3)$$

(3) from (2) gives $3x=27 \ldots\ldots\ldots\ldots x=9.$

(17.) Multiply the 1st equation by 14 and we have

$$42x-7y=49$$

Add $$-x+7y=33$$

$$41x \quad =82 \quad \text{or,} \ldots\ldots x=2.$$

(18.) $$x+\frac{2}{3}y=16$$

$$x-\frac{3y}{5}=-3$$

Subtract $$\frac{2}{3}y+\frac{3}{5}=19$$

$$10y+9y=19 \cdot 15 \ldots\ldots y=15.$$

(19.) Divide 2d by the 1st, and $x-y=2$

But $x+y=8.$

(20.) Multiply the first equation by $(x+y)$, and the second by 9, and we have

$$4(x+y)^2=9(x^2-y^2) \qquad 9(x^2-y^2)=9 \cdot 36$$

$$4(x+y)^2=9 \cdot 36$$

Hence, $x+y=9$

Divide the 1st equation by this last, and we have

$$x-y=4.$$

(21.) $$x=\frac{4y}{3} \qquad x^3=\frac{64y^3}{27}$$

$$\frac{64y^3}{27}-y^3=37$$

$$37y^3=37\cdot 27 \cdots\cdots\cdots\cdots y=3.$$

(22) and (23) require no remark.

(24.) The first equation gives

$$x+24y=\ 91 \qquad (1)$$

Add $$40x+\ y=763 \qquad (2)$$

Multiply (1) by 40, and subtract equation (2), and

$$959y=2877 \quad \text{or,} \cdots\cdots\cdots\cdots y=3.$$

(25.) From 1st equation take the 2d, and we have

$$2\tfrac{1}{2}x+5y=60.$$

Divide by $2\frac{1}{2}$ and we have $x+2y=24$

But $\frac{1}{2}x+2y=19$

$\frac{1}{2}x \quad = 5$ or, $x=10$.

(26.) Add the two equations, and

$$\tfrac{1}{2}(x+y)+\tfrac{1}{3}(x+y)+20=x+y$$

or, $$\frac{s}{2}+\frac{s}{3}+20=s \cdots\cdots\cdots\cdots s=120.$$

By 2d equation, $\frac{1}{3}(120)-5=y=35$, *Ans.*

CHAPTER III.

EQUATIONS OF THREE OR MORE UNKNOWN QUANTITIES.

(7.) Given $\left\{\begin{array}{l} 2x=u+y+z \\ 3y=u+x+z \\ 4z=u+x+y \\ u=x-14 \end{array}\right\}$ to find u, x, y and z.

Subtract 2d from the 1st, and

$$2x-3y=y-x \quad \text{or} \quad 3x=4y \qquad (1)$$

Subtract 2d from the 3d, and

$$4z-3y=y-z \quad \text{or} \quad 5z=4y \qquad (2)$$

Add 3d and 4th and $4z=2x+y-14$ (3)

Multiply (3) by 5 and (2) by 4 and

$$\begin{array}{r} 202=10x+5y-70 \\ 202=\qquad 16y \\ \hline 0=10x-11y-70 \end{array}$$

or $30x-33y=210$

But, equation (1) $30x-40y=\ \ 0$

$$\begin{array}{r} 7y=210 \\ y=\ 30 \end{array}$$

(8.) $$\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62 \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47 \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38 \end{array}\right\} \begin{array}{l} =a+b \\ =a-\frac{b}{4} \\ =a-b \end{array}$$

To avoid numeral multiplication, and really, to understand *algebra* as applied here, observe that $62+38=100$.

Put $a=50$; then $62=a+12=a+b$.

Clearing of fractions, we have

$$6x+4y+3z=12a+12b \qquad (1)$$
$$20x+15y+12z=60a-15b \qquad (2)$$
$$15x+12y+10z=60a-60b \qquad (3)$$

Multiply (1) by 4, and subtract (2).

Then, $4x+y=63b-12a$ (4)

Subtract (3) from (2,) and

$$\begin{array}{r} 5x+3y+2z=\ 45b \\ 3\qquad\qquad\qquad \\ \hline 15x+9y+6z=135b \end{array}$$

Subtract $12x+8y+6z=\ 24b+24a$

$$3x+\ y\qquad =111b-24a \qquad (5)$$

Subtract (5) from (4), and we have

$x=(12a-48b)=12(a-4b)=12\cdot2$, *Ans.*

That is, $x=24$ or $2b$.

Now, equation (4) gives us

$$8b+y=63b-ba$$
$$y=(55-a)b=5\cdot 12=60.$$

(9.) By adding the three equations and reducing, we have

$$4x+3y+2z=3a \qquad (1)$$

By adding the 2d and 3d, reducing and doubling, we have

$$10x+4y+2z=4a \qquad (2)$$

Subtracting (1) from (2), and we have

$$6x+y=a \qquad (3)$$

Adding the 1st and 3d, and reducing, we have

$$x+2y=a \qquad (4)$$

From (3) and (4) we readily find x and y.

$$(10.) \quad \begin{cases} 2x+y-2z=40 & (1) \\ 4y-x+3z=35 & (2) \\ 3u+t=13 & (3) \\ y+u+t=15 & (4) \\ 3x-y+3t-u=49 & (5) \end{cases}$$

It is easier to eliminate t than any other letter.

Subtract (3) from (4), and we have

$$y-2u=2 \qquad (6)$$

Three times (3) taken from (5), gives

$$3x-y-10u=10 \qquad (7)$$

Add (6) and (7) and divide the sum by 3, and

$$x-4u=4 \qquad (8)$$

Double (6), and subtract it from (8,) and we have

$$x=2y \qquad (9)$$

Eliminate z from equations (1) and (2), and we have

$$4x+11y=190$$

But $4x=8y$. Then $19y=190$, or $\cdots\cdots\cdot y=10$.

PROBLEMS PRODUCING SIMPLE EQUATIONS, INVOLVING TWO OR MORE UNKNOWN QUANTITIES.

(**1.**) Let x, y, and z represent the numbers.

$$xy=600 \qquad (1)$$
$$xz=300 \qquad (2)$$
$$yz=200 \qquad (3)$$

Multiply (1) and (2), and divide the product by equation 3, and we have $x^2=900$ . . . . $x=30$.

(**2.**) Let x, y, and z represent the numbers. Then, per question

$$x+\frac{y}{2}+\frac{z}{2}=120 \qquad (1)$$

$$y+\frac{x-z}{5}=70 \qquad (2)$$

$$x+y+z=190 \qquad (3)$$

Double (1) and subtract (3), gives . . . . $x=50$.

This problem calls the pupil's judgment into exercise. He does not know in the first place which is greatest, x or z; hence he must try both suppositions, and the one that verifies equation (2) is right.

(**3.**) Let x, y, and z represent the shares, and put $a=120$

$$x-\tfrac{4}{7}(y+z)=a$$
$$y-\tfrac{3}{8}(x+z)=a$$
$$z-\tfrac{2}{9}(x+y)=a$$

Clearing of fractions, we have

$$7x-4y-4z=7a \qquad (1)$$
$$-3x+8y-3z=8a \qquad (2)$$
$$-2x-2y+9z=9a \qquad (3)$$

Double (1) and to the product add (2,) and we have

$$11x-11z=22a$$
$$x-z=2a \qquad (4)$$

Double (3), and to the product add (1), and we have

$$-11x+22z=11a$$

$$-x+2z=a \qquad (5)$$

Add (4) and (5), and we have $\cdots\cdots z=3a=360.$

(4.) Resolved in the book.

Let x, y, u, and z represent their ages, and s their sum.

Then,

$$\begin{aligned} s-z&=18 \\ s-u&=16 \\ s-y&=14 \\ s-x&=12 \\ \hline 3s\quad&=60 \end{aligned}$$

By addition, $3s=60 \cdots\cdots\cdots\cdots s=20.$

(5.) Let x=A's shillings.

y=B's "

z=C's "

After the first game they will have as here represented:

$$\begin{aligned} x-y-z&=A \\ 2y\qquad&=B \\ 2z\qquad&=C \end{aligned}$$

After the second game,

$$\begin{aligned} 2x-2y-2z&=A \\ 3y-x-z&=B \\ 4z\qquad&=C \end{aligned}$$

After the third game

$$\begin{aligned} 4x-4y-4z&=16 &(1) \\ 6y-2x-2z&=16 &(2) \\ 7z-x-y&=16 &(3) \\ \hline x+y+z&=3\cdot16 &(4) \end{aligned}$$

Sum, $x+y+z=3\cdot16 \qquad (4)$

Add (3) and (4) and we have

$8z=4\ 16 \cdots\cdots\cdots\cdots\cdots\cdots z=8.$

(6.) This problem is resolved in the book, by equation 7, Art. 53.

(7.) Let x represent the better horse, and y the poorer.

$$x+15=\tfrac{4}{3}(y+10)$$

$$x+10=\tfrac{15}{13}(y+15)$$

Therefore, $\tfrac{4}{3}(y+10)=\tfrac{15}{13}(y+15)+5$

Reduced gives . . . . $y=50$.

(8.) Let $x=$ the price of the sherry.

$y=$ . . . . brandy.

Put $a=78$.

$$\begin{array}{l} 2x+\ y=3a \\ 7x+2y=9a+9 \\ \hline 3x\qquad=3a+9 \\ x\qquad=\ a+3= \end{array}$$ . . 81 *s.*, *Ans.*

(9.) Let $x=$A's time. $y=$B's time.

Then, $\frac{1}{x}=$the part that A can do in one day.

And $\frac{1}{y}=$the part that B can do in one day.

$$\frac{4}{x}+\frac{4}{y}=\frac{4}{16}=\frac{1}{4}$$

$$\frac{36}{y}=\frac{3}{4}$$ Hence . . . . . $y=48$.

(10.) $\frac{2x}{y+7}=\frac{2}{3}$ $\qquad$ $\frac{x+2}{2y}=\frac{3}{5}$

$3x=y+7$ $\qquad$ $5x+10=6y$.

(11.) $x+\frac{2y}{3}=a$ $\qquad$ $y+\frac{3x}{4}=a$

(12.) Let $x=$ the greater, and $(24-x)=$ less.

$$\frac{x}{24-x} : \frac{24-x}{x} :: 4 : 1$$

$$x^2 : (24-x)^2 :: 4 : 1$$

By evolution, $x : 24-x :: 2 : 1$.

(**13.**) Let $x=$ the number of persons.

$y=$ what each had to pay.

Then, $xy=$ the amount of the bill.

Put $(x+4)(y-1)=$ . . . . the bill.

Also, $(x-3)(y+1)=$ . . . . the bill.

$$xy+4y-x-4=xy$$
$$xy-3y+x-3=xy$$

$$4y-x-4=0$$
$$-3y+x-3=0$$

By addition, $y \quad -7=0$

(**14.**) $10x+y=4x+4y$ or, ... $y=2x$.

$$10x+y+27=10y+x$$

or, $9x \quad +27= \ 9y$

$x \quad + \ 3= \quad y=2x$ hence, $x=3$.

(**15.**) Let $x=$ the digits in the place of 100's.

$y=$ " in the place of 10's.

$z=$ " the units.

$x+y+2=11$ $z=2x$

$$100x+10y+z+297=100z+10y+x$$
$$99x+297= \ 99z$$

$x+3=z=2x$ Hence . . . . $x=3$.

(**16.**) Let $\frac{x-40}{2}$, $\frac{x-20}{3}$, and $\frac{x-10}{4}$ represent the parts. Then

$$\frac{x-40}{2}+\frac{x-20}{3}+\frac{x-10}{4}=90 \ldots\ldots x=100.$$

(**17.**) Let x represent the part at 5 per cent, and $(a-x)$ the part at 4 per cent. Then

$$\frac{5x}{100}+\frac{4a-4x}{100}=b$$

Hence . . . . $x=100b-4a$.

(**18.**) To avoid high numerals, and of course a tedious operation, Put $a=5000$; then $2a=10000$, $3a=15000$,

$$\frac{3a}{10}=1500, \text{ and } \frac{16a}{100}=800.$$

Put $x=$ A's capital, and $r-1=$ A's rate.
$x+2a=$ B's " $r=$ B's rate.
$x+3a=$ C's " $r+1=$ C's rate.

$$\text{By conditions,}\begin{cases}\dfrac{rx-x}{100}+\dfrac{16a}{100}=\dfrac{rx+2ar}{100}\cdots\cdots\cdots (1)\\[2ex] \dfrac{rx-x}{100}+\dfrac{3a}{10}=\dfrac{rx+3ar+x+3a}{100}\cdots (2)\end{cases}$$

Reducing (1), gives $x=(16-2r)a$

" (2), " $x=\left(\dfrac{27-3r}{2}\right)a$

Hence, $32-4r=27-3r$, or $\cdots\cdots\cdots\cdots r=5$.

(**19.**) Put $a=1000$, x and y to represent the two parts, and r and t the rates expressed in *decimals*.

$$\text{Then by conditions,}\begin{cases}x+y=13a\cdots\cdots\cdots\cdots (1)\\ rx=\ ty\cdots\cdots\cdots\cdots (2)\\ tx=360\cdots\cdots\cdots\cdots (3)\\ ry=490\cdots\cdots\cdots\cdots (4)\end{cases}$$

Divide (3) by (4), and we have

$$\left(\frac{t}{r}\right)\left(\frac{x}{y}\right)=\frac{36}{49}\cdots\cdots\cdots (5)$$

From (2) we have $\dfrac{x}{y}=\dfrac{t}{r}$

Substitute the value of $\dfrac{t}{r}$ in equation (5), and

$$\frac{x^2}{y^2}=\frac{36}{49} \quad \text{or} \quad x=\frac{6y}{7}$$

By returning to equation (1) we have $\dfrac{6y}{7}+y=13a$.

$13y=13a\cdot 7$ or $\cdots\cdots\cdots\cdots\cdots y=7a$.

(**20.**) Let x, y, and z, represent their respective ages.

Then by conditions given, $x-y=z$

$$5y+2z-x=147$$
$$x+y+z=96$$

(**21.**) Let x, y, and z, represent the respective property of each, and put s=their sum.

$$\text{Conditions,}\begin{cases} x+3y+3z=47a \\ y+4x+4z=58a \\ z+5x+5y=63a \end{cases} \qquad a=100.$$

Add $2x$ to the 1st equation, $3y$ to the 2d, and $4z$ to the 3d, observing that $x+y+z=s$; then we shall have

$$3s=47a+2x \ldots\ldots\ldots\ldots (1)$$
$$4s=58a+3y \ldots\ldots\ldots\ldots (2)$$
$$5s=63a+4z \ldots\ldots\ldots\ldots (3)$$

or, $$x=\frac{3s-47a}{2}$$

$$y=\frac{4s-58a}{3}$$

$$z=\frac{5s-63a}{4}$$

By addition, $s=\frac{3s-47a}{2}+\frac{4s-58a}{3}+\frac{5s-63a}{4}$

Hence, $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots s=19a.$

This value of s, put in equation (1), gives $x=5a=500.$

(**22.**) Taken from the book we have

$$s=\frac{ls-p}{l-1}+\frac{ms-q}{m-1}+\frac{ns-r}{n-1}$$

or, $s=\left(\frac{l}{l-1}\right)s-\frac{p}{l-1}+\left(\frac{m}{m-1}\right)s-\frac{q}{m-1}+\left(\frac{n}{n-1}\right)s-\frac{r}{n-1}$

or, $\left(\frac{l}{l-1}+\frac{m}{m-1}+\frac{n}{n-1}-1\right)s=\frac{p}{l-1}+\frac{q}{m-1}+\frac{r}{n-1}$

Hence, $$s=\frac{\frac{p}{l-1}+\frac{q}{m-1}+\frac{r}{n-1}}{\frac{l}{l-1}+\frac{m}{m-1}+\frac{n}{n-1}-1}=k$$

As s is now known, we call its value k.

Then, ... $x=\frac{lk-p}{l-1}$

$$y=\frac{mk-q}{m-1}$$

$$z=\frac{nk-r}{r-1}$$

(**23.**) Let x, y, and z represent the respective sums.

$$x+\frac{y}{2}=a \quad \ldots\ldots\ldots\ldots (1)$$

$$y+\frac{z}{3}=a \quad \ldots\ldots\ldots\ldots (2)$$

$$z+\frac{x}{4}=a \quad \ldots\ldots\ldots\ldots (3)$$

$$\begin{array}{lll}
2x+y=2a & & \\
3y+z=3a & \text{or,} & 4z+12y=12a \\
4z+x=4a & \text{or,} & 4z+x=\ 4a \\
\hline
 & & -x+12y=\ 8a \\
\text{From the 1st} & & 24x+12y=24a \\
\hline
 & & 25x\qquad\ =16a
\end{array}$$

(**24.**) This problem is resolved in the work, by the 13th example, page 80, (Art 51.)

(**25.**) Let $x=$ the greater, and y the less.

$\frac{1}{2}x+\frac{1}{3}y=13$

$\frac{1}{3}x-\frac{1}{2}y=0$ or, ... $2x=3y$.

(26.)

$$x+\tfrac{1}{2}(y+z)=a=51$$
$$y+\tfrac{1}{3}(x+z)=a$$
$$z+\tfrac{1}{4}(x+y)=a$$
$$x+(x+y+z)=2a$$
$$2y+(x+y+z)=3a$$
$$3z+(x+y+z)=4a$$

$$x=2a-s \cdots\cdots\cdots (1)$$
$$y=\tfrac{1}{2}(3a-s) \cdots\cdots\cdots (2)$$
$$z=\tfrac{1}{3}(4a-s) \cdots\cdots\cdots (3)$$

$$s=2a-s+\tfrac{1}{2}(3a-s)+\tfrac{1}{3}(a-s)$$
$$6s=12a-6s+9a-3s+8a-2s$$
$$17s=29a \quad \text{or} \cdots\cdots\cdots s=29\cdot3=87.$$

Now equations (1), (2), and (3), will readily give x, y, and z.

(27.) Let x=A's, y B's, and z=C's sheep.

Then by the conditions,

$$x+8-4=y+z-8$$
$$\tfrac{1}{2}(y+8)=x+z-8$$
$$\tfrac{1}{3}(z+8)=x+y-8$$

$$x+12=y+z \cdots\cdots\cdots (1)$$
$$y+24=2x+2z \cdots\cdots\cdots (2)$$
$$z+32=3x+3y \cdots\cdots\cdots (3)$$

Add (1) and (3), and we have $x+44=3x+4y$.

Double (1), and subtract (2), and we have

$$2x-y=2y-2x \qquad \text{or,} \qquad 4x=3y$$

But $\cdots\cdots\cdots 2x+4y=44$

$$4x+8y=44\cdot2$$
$$11y=44\cdot2 \quad \text{or} \cdots\cdots y=8.$$

(28.) $\dfrac{x+1}{y}=\dfrac{1}{3}$ $\qquad$ $\dfrac{x}{y+1}=\dfrac{1}{4}$

(29.) $\dfrac{x+2}{y}=\dfrac{5}{7}$ $\qquad$ $\dfrac{x}{y+2}=\dfrac{1}{3}$

(30.) This is a repetition of the 10th example, page 89, inserted here by oversight.

(31.) Let x=A's money, and y=B's.

$$x-5=\tfrac{1}{2}(y+5)\cdots\cdots\cdots(1)$$
$$x+5=3y-15\cdots\cdots\cdots(2)$$

Subtract (1) from (2), and we have

$$10=3y-15-\frac{y}{2}-\frac{5}{2}\quad\text{or}\cdots\cdots\cdots y=11.$$

(32.) Let $x=$ the number of bushels of wheat flour.
And $y=\cdots\cdots\cdots$ barley "
Then the cost of the whole will be expressed by

$$10x+4y$$

The sale at 11 shillings will be $11x+11y$
Now by the conditions,

$$10x+4y\ :\ 11x+11y\ ::\ 100\ :\ 143\tfrac{3}{4}$$

Multiply the last two terms by 4, and

$$10x+4y\ :\ 11x+11y\ ::\ 400\ :\ 575$$

Divide the two last terms by 25, and

$$10x+4y\ :\ 11x+11y\ ::\ 16\ :\ 23$$
$$5x+2y\ :\ 11x+11y\ ::\ 8\ :\ 23$$
$$115x+46y=88x+88y$$
$$27x=42y$$
$$9x=14y$$

These co-efficients, 9 and 14, give the lowest proportion in whole numbers. The proportion was only required.

(33.) Let $10x+y$ represent the number.

$$\tfrac{1}{5}(10x+y)=Q+\tfrac{1}{5}$$
$$\tfrac{1}{8}(10x+y)=Q'+\tfrac{1}{8}$$

Now the question gives us $Q=2x$
And $\cdots\cdots\cdots Q'=5y$

$$\tfrac{1}{5}(10x+y)=2x+\tfrac{1}{5}\quad\text{or}\cdots\cdots\cdots y=1.$$

INTERPRETATION OF NEGATIVE VALUES.

(Art. 55.)

(**4.**) Let x represent the years to elapse.

Then $30+x=3(15+x)$ . . . . $x=-7\frac{1}{2}$.

To make this equation true, the years required must be taken *subtractively.*

(**5.**) Let $x=$ the man's daily wages, and $y=$ the son's.

$$7x+3y=22 \quad \ldots\ldots\ldots\ldots (1)$$
$$5x+\ y=18 \quad \ldots\ldots\ldots\ldots (2)$$
$$12x+4y=40$$
$$3x+\ y=18 \qquad x=4,\quad y=-2.$$

(**6.**) Let $x=$ man's wages, $y=$ wifes, and $z=$ the son's

$$10x+\ 8y+\ 6z=1030 \text{ cts.} \ldots\ldots (1)$$
$$12x+10y+\ 4z=1320 \text{ cts.} \ldots\ldots (2)$$
$$15x+10y+12z=1385 \text{ cts.} \ldots\ldots (3)$$

Subtract (2) from (3), and we have

$$3x+8z=65 \quad \ldots\ldots\ldots\ldots (4)$$

Multiply (1) by (5), and (2) by 4, and take their difference, and we have $2x+14z=\ 130$

$$x+\ 7z=-\ 65 \quad \ldots\ldots\ldots (5)$$
$$3x+21z=-3\cdot 65$$
$$(4) \ldots\ldots\ldots\ldots 3x+\ 8z=\ 65$$

By subtraction, $13z=-4\cdot 65$

$$z=-4\cdot 5=-20$$

As z comes out with a *minus* value, it shows that the son had no wages, but the reverse of it, he was on expense.

(**7.**)

$$10x+4y+3z=1150 \quad \ldots\ldots\ldots (1)$$
$$9x+8y+6z=1200 \quad \ldots\ldots\ldots (2)$$
$$7x+6y+4z=\ 900 \quad \ldots\ldots\ldots (3)$$

Double (1), and subtract (2), and

$$11x=1100 \quad \ldots\ldots\ldots\ldots\ldots\ldots x=100 \text{ cts.}$$

(8.) $\frac{x+1}{y}=\frac{3}{5}$ $\qquad$ $\frac{x}{y+1}=\frac{5}{7}$

$$5x+5=3y \cdots\cdots\cdots (1)$$
$$7x \quad =5y+5 \cdots\cdots (2)$$

Add $\cdots\cdots\cdots$ $12x \quad =8y$

$$3x \quad =2y \cdots\cdots\cdots (3)$$

Subtract (2) from (1) and we have

$$-2x+5=-27-5 \quad \cdots\cdots (4)$$

$$x=-10 \text{ by adding (3) and (4).}$$
$$y=-15$$

The result coming out *minus*, shows that there is no such arithmetical fraction. Algebraically, however, $\frac{-10}{-15}$ will answer the conditions.

SECTION III.

We pass over the first three chapters of this section, as requiring no aid from a key.

We commence at the examples in approximate cube root at page 126.

(1.) What is the approximate cube root of 120 ?

125 has a root of 5.

2						
250		240				
120		125				
370	:	365	: :	5	:	root required.

(2.) What is the approximate cube root of 8.5 ?

8 has a cube root of 2.

16		17				
8.5		8				
24.5	:	25	: :	2	:	root required.

(3.) What is the approximate cube root of 63?

64 has a cube root of 4.

128		126				
63		64				
191	:	190	: :	4	:	root required.

(4.) What is the approximate cube root of 515?

512 has a cube root of 8.

	1024		1030				
	515		512				
	1539	:	1542	: :	8	:	cube required.
or	1539	:	3	: :	8	:	correction.

SURDS IN GENERAL.

(2.) $\sqrt{98a^2} = \sqrt{49a^2 \times 2} = 7a\sqrt{2}$

(3.) $\sqrt{12x^2y} = \sqrt{4x^2 \times 3y} = 2x\sqrt{3y}$

(4.) $\sqrt[3]{54x^4} = \sqrt[3]{27x^3 \times 2x} = 3x\sqrt[3]{2x}$

(5.) $4\sqrt[3]{108} = 4\sqrt[3]{27. \times 4} = 12\sqrt[3]{4}$

(6.) $\sqrt{x^3 - a^2x^2} = \sqrt{x^2(x - a^2)} = x\sqrt{x - a^2}$

(7.) $\sqrt[3]{32a^3} = \sqrt[3]{8a^3 \times 4} = 2a\sqrt[3]{4}$

(8.) $\sqrt{28a^3x^2} = \sqrt{4a^2x^2 \times 7a} = 2ax\sqrt{7a}$

(9.) Performed in the work.

(10.) $\sqrt[3]{\frac{135}{64}} = \sqrt[3]{\frac{270}{64}} = \sqrt[3]{\frac{27}{64} \times 10} = \frac{3}{4}\sqrt[3]{10}$

(11.) $\sqrt[3]{\frac{25}{9}} = \sqrt[3]{\frac{75}{27}} = \sqrt[3]{\frac{1}{27} \times 75} = \frac{1}{3}\sqrt[3]{75}$

(12.) $\sqrt{\frac{50}{147}} = \sqrt{\frac{25 \cdot 2}{49 \cdot 3}} = \sqrt{\frac{25 \cdot 6}{49 \cdot 9}} = \frac{5}{21}\sqrt{6}$

(**13.**) $\sqrt[3]{a^3+a^3b^2}=\sqrt[3]{a^3(1+b^2)}=a\sqrt[3]{1+b^2}$

(**14.**) $$\sqrt{\frac{2a}{3}}=\sqrt{\frac{1}{9}\times 6a}=\frac{1}{3}\sqrt{6a}$$

(Art. 81.) Requires no aid from a key.

(Art. 82.)

(**1.**) $5\sqrt{5}\times 3\sqrt{8}=15\sqrt{5\times 2\times 4}=30\sqrt{10}$

(**2.**) $4\sqrt{12}\times 3\sqrt{2}=12\sqrt{24}=24\sqrt{6}$

(**3.**) $3\sqrt{2}\times 2\sqrt{8}=6\sqrt{16}=24$

(**4.**) $2\sqrt[3]{14}\times 3\sqrt[3]{4}=6\sqrt[3]{7\times 8}=12\sqrt[3]{7}$

(**5.**) $2\sqrt{5}\times 2\sqrt{10}=4\sqrt{50}=20\sqrt{2}$

(**1.**) Here to multiply $(a+b)^{\frac{1}{3}}$ by $(a+b)^{\frac{3}{4}}$ we must simply *add the exponents.*

$$(a+b)^{\frac{1}{3}+\frac{3}{4}}=(a+b)^{\frac{13}{12}}$$

(**2.**) $(7)^{\frac{1}{2}}\times 7^{\frac{1}{3}}=7^{\frac{1}{2}+\frac{1}{3}}=7^{\frac{5}{6}}$

(**3.**) $2(3)^{\frac{1}{2}}\times 3(4)^{\frac{1}{3}}=2(3)^{\frac{3}{6}}\times 3(4)^{\frac{2}{6}}=2(27)^{\frac{1}{6}}\times 3(16)^{\frac{1}{6}}=$ $6(27\times 16)^{\frac{1}{6}}=6(432)^{\frac{1}{6}}$

(**4.**) $(15)^{\frac{1}{3}}\times(10)^{\frac{1}{2}}=(15)^{\frac{2}{6}}\times(10)^{\frac{3}{6}}=(225)^{\frac{1}{6}}\times(1000)^{\frac{1}{6}}=$ $(225000)^{\frac{1}{6}}$

PURE EQUATIONS.

(**14.**) Multiply both members by $\sqrt{a+x}$, and

$$\sqrt{ax+x^2}+a+x=2a$$
$$\sqrt{ax+x^2}=a-x$$
$$ax+x^2=a^2-2ax+x^2$$
$$3ax=a^2 \;\ldots\ldots\ldots\ldots\; x=\tfrac{1}{3}a.$$

(**15.**) Multiply by $\sqrt{a^2+x^2}$, then

$$x\sqrt{a^2+x^2}+a^2+x^2=2a^2$$
$$x\sqrt{a^2+x^2}=a^2-x^2$$

By squaring both members and reducing we have

$3a^2x^2=a^4$ or . . . . $x=\pm a\sqrt{\tfrac{1}{3}}$.

(**16.**) Square both members, and

$$x^2+2ax+a^2=a^2+x\sqrt{b^2+x^2}$$

Drop a^2 and divide by x, and

$$x+2a=\sqrt{b^2+x^2} \qquad \text{Square, \&c.}$$

(**17.**) Divide the numerator by its denominator, in each member, and we have

$$1-\frac{4}{\sqrt{6x}+2}=1-\frac{15}{4\sqrt{6x}+6}$$

Drop 1 and change signs, and clear of fractions, and

$16\sqrt{6x}+24=15\sqrt{6x}+30$ Hence . . . . . $x=6$.

(**18.**) Cube both members, and

$$64+x^2-8x=\frac{(4+x)^3}{(4+x)}=(4+x)2$$

Hence, $64+x^2-8x=16+8x+x^2$ or . . . . . $x=3$.

(**19.**) By clearing of fractions, $5+x+\sqrt{x^2+5x}=15$

By reduction, $\sqrt{x^2+5x}=10-x$ Square &c.

(**20.**) $$\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\frac{3}{2}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x}}}$$

Multiply by $\sqrt{x+\sqrt{x}}$, and we have

$$x+\sqrt{x}-\sqrt{x^2-x}=\tfrac{3}{2}\sqrt{x}$$
$$2x+2\sqrt{x}-2\sqrt{x^2-x}=3\sqrt{x}$$
$$2x-\sqrt{x}=2\sqrt{x^2-x}$$
$$4x^2-4x\sqrt{x}+x=4x^2-4x \qquad x=\tfrac{25}{16}.$$

(**21.**) Resolved in the work.

(**22.**) Resolved the same as 17.

$$1-\frac{2b}{\sqrt{ax}+b}=1-\frac{7b}{3\sqrt{ax}+5b}$$

Hence, $6\sqrt{ax}+10b=7\sqrt{ax}+7b$ or . . . . . $x=\frac{9b^2}{a}$

(*U.* 136.) (*S.* 136.)

(**23.**) Square both members and we have

$$\cancel{1}+x\sqrt{x^2+12}=\cancel{1}+2x+x^2$$

$\sqrt{x^2+12}=2+x$ Square, &c. $x=2.$

(**24.**) Multiply numerator and denominator by the numerator, and we have

$$\frac{(\sqrt{4x+1}+\sqrt{4x})^2}{1}=9$$

Take square root, and transpose $\sqrt{4x}$, and

$$\sqrt{4x+1}=3-\sqrt{4x}$$

Square, $4x+1=9-6\sqrt{4x}+4x$ Hence, $x=\frac{4}{9}$.

(**25.**) Square root gives $a-x=\sqrt{b}$ or, $x=a-\sqrt{b}$.

(**26.**) Clearing of fractions and we have

$$1+\sqrt{1-x^2}-1+\sqrt{1-x^2}=\sqrt{3}$$

$$2\sqrt{1-x^2}=\sqrt{3}$$

$$4-4x^2=3 \qquad x^2=\tfrac{1}{4}$$

$$x=\pm\tfrac{1}{2}.$$

(**27.**) Take the square root of both members, and

$$\frac{2}{x-1}=\frac{1}{2} \text{ or } \ldots\ldots\ldots\ldots 4=x-1$$

(**28.**) Resolved the same as the 21st.

(**29.**) Clearing of fractions we have

$$\sqrt{x^2-9x}+x-9=36$$

$$\sqrt{x^2-9x}=45-x$$
$$x^2-9x=45^2-90x+x^2$$
$$81x=45\cdot45$$
$$9\cdot9x=5\cdot9\cdot5\cdot9\ldots\ldots x=5\cdot5=25.$$

(**30.**) Resolved the same as (**17**) and (**22.**)
Dividing numerators by denominators, we have

$$3-\frac{10}{\sqrt{x}+2}=3-\frac{10\cdot5}{\sqrt{x}+40}$$

Drop 3 from both sides, change signs, and divide by 5, and clear of fractions, and

$$2\sqrt{x}+80=21\sqrt{x}+42 \quad \text{Hence} \ldots\ldots\ldots x=4.$$

(**31.**) Multiply numerator and denominator of the first member by $(\sqrt{x}+\sqrt{x-a})$

Then, $$\frac{(\sqrt{x}+\sqrt{x-a})^2}{a}=\frac{an^2}{x-a}$$

Multiply by a, and take square root, and

$$\sqrt{x}+\sqrt{x-a}=\frac{an}{\sqrt{x-a}}$$
$$\sqrt{x^2-ax}+x-a=an$$
$$\sqrt{x^2-ax}=(n+1)a-x$$
$$x^2-ax=(n+1)^2a^2-2a(n+1)x+x^2$$

Drop x^2, and divide by a, and

$$-x=(n+1)^2a-2nx-2x$$
$$(1+2n)x=(n+1)^2a$$
$$x=\frac{(n+1)^2a}{1+2n}$$

(**32.**) Resolved in the work.

(Art. 90.)

(**4.**) Observe $180=9\cdot20 \quad 189=9\cdot21$. Put $a=9$.

$$x^2y+xy^2=20a \qquad x^3+y^3=21a$$

Multiply the first equation by 3, and add it to the second,

and $x^3+3x^2y+3xy^2+y^3=81a=a^3$

cube root, $x+y=a=9$

The rest of the operation is obvious

(**5.**) Divide the first equation by $(x+y)$ and

$$x^2-xy+y^2=xy$$

or $x^2-2xy+y^2=0$ or $x-y=0$.

Hence . . . . $x=2$ $y=2$

(**6.**) $x+y : x :: 7 : 5$ $xy+y^2=126$

$$5x+5y=7x$$

$$5y=2x \quad \text{or} \quad x=\tfrac{5}{2}y.$$

Put this value of x in the second equation, and

$$\tfrac{5}{2}y^2+y^2=126$$

$$7y^2=126\cdot2$$

$$y^2=18\cdot2=36 \quad \ldots \quad y=\pm6.$$

(**7.**) From the first equation we have

$$5x-5y=4y \qquad 5x=9y$$

$$x=\tfrac{9}{5}y$$

$$x^2+4y^2=181$$

$$\tfrac{81}{25}y^2+4y^2=181$$

$$81y^2+100y^2=181\cdot25$$

$181y^2=181\cdot25$ or . . . . $y^2=25$.

(**8.**) From the proportion we have

$$\sqrt{x}+\sqrt{y}=4\sqrt{x}-4\sqrt{y}$$

$5\sqrt{y}=3\sqrt{x}$ or . . . . $25y=9x$.

The rest of the operation is obvious.

(**9.**) Extract square root and

$\tfrac{1}{3}x+\tfrac{1}{2}=3$ or . . . . $x=7\tfrac{1}{2}$.

(**10.**) From the first proportion

$x+y=3x-3y$ or $4y=2x$

Hence . . . . $8y^3=x^3$.

$8y^3-y^3=56$ $y^3=8$ $y=2$.

(**11**) (**12**) and (**13**) resolved in the work.

(**14.**) $\frac{x^2-y^2}{x-y}=6$ or . . . . $x+y=6$

From the second, ... $xy=5.$

(**15.**) Divide the 1st equation by $(x+y)$, and

$$x^2-\ xy+y^2=2xy$$

$$x^2-2xy+y^2=xy=16 \ \ldots\ldots (1)$$

Add $\qquad 4xy \qquad =64$

$$x^2+2xy+y^2=\quad 80=16\cdot 5$$

Square root, $\qquad x+y=4\sqrt{5}$

Square root of (1) $\quad x-y=4$

$$2x \quad =4\sqrt{5}+4$$

(**19.**) Double the 2d equation, and add and subtract it from the 1st, then

$$x^2+2xy+y^2=a+2b$$

$$x^2-2xy+y^2=a-2b$$

$$x+y=\sqrt{a+2b}$$

$$x-y=\sqrt{a-2b}$$

(Art. 92.)

(**5.**) Add the two equations and extract square root, and we have $\qquad x^{\frac{1}{2}}+y^{\frac{1}{2}}=\pm 4$ . . . . (1)

Separate the first member of the first equation into factors, and we have $\qquad x^{\frac{1}{2}}\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)=12$ . . . . . . (2)

Divide (2) by (1) and $\quad x^{\frac{1}{2}}=\pm 3$ . . . . $x=9.$

(**6.**) is of the same form and resolved the same as (**5.**)

(**7.**) Add the two equations, and extract square root, and we have $\qquad x^{\frac{3}{4}}+y^{\frac{3}{4}}=\sqrt{a+b}$

But . . . . $x^{\frac{3}{4}}(x^{\frac{3}{4}}+y^{\frac{3}{4}})=a$

$$x^{\frac{3}{4}}=\frac{a}{\sqrt{a+b}}$$

$$x^3=\frac{a^4}{(a+b)^2} \quad \text{or} \ldots\ldots\ldots x=\left(\frac{a^4}{(a+b)^2}\right)^{\frac{1}{3}}$$

(8.) Resolved in (Art. 90,) of the Key.

(9.) Square the first equation, and

$$x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y=25$$

$$x \qquad +y=13 \ldots\ldots\ldots(1)$$

Difference . . . . . . $2x^{\frac{1}{2}}y^{\frac{1}{2}} \quad =12 \ldots\ldots\ldots(2)$

Subtract (2) from (1) and

$$x-2x^{\frac{1}{2}}y^{\frac{1}{2}}+y=1$$

By evolution . . . . . . $x^{\frac{1}{2}}-y^{\frac{1}{2}}=\pm 1$

But . . . . $x^{\frac{1}{2}}+y^{\frac{1}{2}}=5$

$$2x^{\frac{1}{2}} \qquad =6 \text{ or } 4.$$

CHAPTER V.

(Art. 93.)

QUESTIONS PRODUCING PURE EQUATIONS.

(5.) Let $x+y=$ the greater number,

And . . . . . . $x-y=$ the less

Difference . . . . $2y=4$ Sum $=2x$

$2x(4xy)=1600$ Hence . . . . $x=10.$

(6.) Let $x+y=$ the greater number.

$x-y=$ the less.

$2y : x-y :: 4 : 3$ or . . . . $x=\frac{5}{2}y.$

$$(x^2-y^2)(x-y)=504$$

$$(\tfrac{25}{4}y^2-y^2)(\tfrac{5}{2}y-y)=504$$

$(\frac{21}{4}y^2)(\frac{3}{2}y) \quad =504$ Hence $y=4.$

(7.) Let $8x=$ the length of the field, and $5x=$ its breadth

Then .... $\frac{40x^2}{160}=\frac{x^2}{4}=$ the acres.

$\frac{1}{4}x^2\times 8x=$the whole cost.

$26x=$ the rods around the field.

$13\times 26x=$the whole cost.

Hence $2x^3=13\cdot 26x$ or $x=13$ ... $8x=104$, *Ans.*

(8.) Let $5x=$ the length of the stack.

$4x=$ the breadth.

Then, $\frac{7x}{2}=$ the height.

$5x\cdot 4x\cdot\frac{7x}{2}\cdot 4x=$ the cost in cents.

Also, $5x\cdot 4x\cdot 224=$ the cost in cents.

Hence, $5x\cdot 4x\cdot\frac{7x}{2}\cdot 4x=5x\cdot 4x\cdot 224$

$7x\cdot 2x=224$ or .... $x=4$.

(9.) Put $x^2-7=$ the number.

Then $x+\sqrt{x^2+9}=9$

$\sqrt{x^2+9}=9-x$ or .... $x=4$.

(10.) Let x represent A's eggs; then $100-x=$ B's eggs.

$\frac{18}{100-x}=$ A's price. $\frac{8}{x}=$B's price.

Hence $\frac{18x}{100-x}=\frac{8}{x}(100-x)$

$9x^2=4(100-x)^2$

$3x=2(100-x)$ .... $x=40$.

(11.) Let $x+y=$ the greater number,

$x-y=$ the less.

$2x=6$ or $x=3$

$$\begin{array}{l}(x+y)^3=x^3+3x^2y+3xy^2+y^3\\ (x-y)^3=x^3-3x^2y+3xy^2-y^3\\ \hline 2x^3 \qquad\quad +6xy^2=72\end{array}$$

Divide by $2x$ and $x^2+3y^2=12$ Hence . . . $y=1$.

(**12.**) Let $x=$ one number

Then $a^2x=$ the other.

$$\overline{a^2x^2=b^2} \qquad ax=b \qquad x=\frac{b}{a}.$$

(**13.**) Let x^2 and y^2 represent the numbers.

Then . . . . $x^2+y^2=100$

$$x+y=14$$

SECTION IV.

QUADRATIC EQUATIONS.

None of the examples require the aid of a key until we come to the 12th, Art. 101.

(**12.**) Put $x=\frac{1}{5}u$. Then $5x^2=\frac{1}{5}u^2$, and the equation becomes . . . . $\frac{1}{5}u^2+\frac{4}{5}u=273$

$$u^2+4u=1365$$

Hence $u=35$ or -39 which gives . . . $x=7$ or $-\frac{39}{5}$.

(**13.**) Put $x=\frac{1}{7}u$. Then

$$u^2-20u \qquad =224$$
$$u^2-20u+100=324$$
$$u-10=\pm 18$$
$$u=28 \text{ or } -8 \quad \text{and} \;.\;.\; x=7 \text{ or } -\tfrac{8}{7}$$

(**14.**) $25x^2-20x=-3$.

Complete the square by (Art. 99.)

$$25x^2-20x+t^2=t^2-3$$

$$5xt=-10x \qquad t=-2 \qquad t^2=4$$

$5x-2=\pm 1$ . . . . $x=\frac{3}{5}$ or $\frac{1}{5}$.

(**15.**) Put $x=\frac{1}{21}u$. Then

$$u^2-292u=-500\cdot21=-10500$$
$$u^2-292n+(146)^2=10816$$
$$u-146=\pm104$$
$$u=250 \text{ or } 42$$

Hence . . . . $x=\frac{250}{21}$ or 2.

(Art. 103.)

EXAMPLES.

(**1.**) Put $(x+12)^{\frac{1}{4}}=y$. Then

$y^2+y=6$ . . . . $y=2$ or -3.

$(x+12)^{\frac{1}{4}}=2$ or -3

$x+12=16$ or 81 . . . . $x=4$ or 69.

(**2.**) Add b^2 to both members and extract square root: we then have $(x+a)^{\frac{1}{4}}+b=\pm2b$

The rest of the operation is obvious.

(**3.**) Put $y=(9x+4)^{\frac{1}{2}}$ Then $y^2+2y=15$

$y=3$ or -5 Hence . . . . $9x+4=9$ or 25.

(**4.**) Put $y=(10+x)^{\frac{1}{4}}$.

Then $y^2-y=2$ . . . . $y=2$ or -1

(Art. 104.)

(**5.**) Put . . . . $(x-5)^{\frac{3}{2}}=y$; then

$y^2-3y=40$ . . . . $y=8$ or -5.

(6.) Put $(1+x-x^2)^{\frac{1}{2}}=y$; then

$$2y^2-y=-\tfrac{1}{9}$$

From which we obtain y equal $\frac{1}{3}$ or $\frac{1}{6}$.

Hence . . $1+x-x^2=\frac{1}{9}$ or $\frac{1}{36}$. From which we obtain

$$x=\tfrac{1}{2}\pm\tfrac{1}{6}\sqrt{41} \text{ or } \tfrac{1}{2}\pm\tfrac{1}{12}\sqrt{176}.$$

(**7.**) Put . . . . $(x+16)^{\frac{1}{2}}=y$

$y^2-3y=10$ Hence $y=5$ or -2.

(8.) Put $y=x^n$; then

$3y^2-2y=8$ Put $y=\frac{1}{3}u$

$u^2-2u=24$ $u=6$ or -4

$y=2$ $x=\sqrt[n]{2}$.

(9.) Multiply by 4, &c. Rule 2.

$$4x^{\frac{6}{5}}+4x^{\frac{3}{5}}+1=3025$$

$$2x^{\frac{3}{5}}+1=\pm 55$$

$$x^{\frac{3}{5}}=27 \text{ or } -28$$

$x^{\frac{1}{5}}=3$ or . . . . $x=243$.

(10.) Put $(2x-4)^2=y$.

Then . . . . $\dfrac{8}{y}=1+\dfrac{16}{y^2}$

$$8y=y^2+16$$

$y^2-8y+16=0$ or . . . . $y-4=0$.

(11.) Multiply by 16. Rule 2.

Then . . . $64x^{\frac{1}{3}}+16x^{\frac{1}{6}}+1=39\cdot16+1=625$

$8x^{\frac{1}{6}}+1=25$ $x^{\frac{1}{6}}=3$ $x=729$.

(12.) Add 5 to both members.

Then $(x^2-2x+5)+6(x^2-2x+5)^{\frac{1}{2}}=16$

By subtraction, $y^2+6y+9=25$ . . . . . $y=2$ or -8.

Hence $x^2-2x+5=4$ . . . . $x=1$.

(13.) By (Art. 99) we have

$$\frac{x^2}{361}-\frac{12x}{19}+t^2=t^2-32$$

$\dfrac{x}{19}t=-\dfrac{6x}{19}$ $t=-6$ $t^2=36$.

$$\frac{x^2}{361}-\frac{12x}{19}+36=4$$

$\dfrac{x}{19}-6=\pm 2$ . . . . $x=152$ or 76.

(**14.**) Observe that $81x^2$ and $\frac{1}{x^2}$ are both squares, and if these are taken for the first and last terms of a binomial square, the middle term must be $9x \cdot \frac{1}{x} \cdot 2 = 18$.

This indicates to add $1 \cdot$ to both members. Then extract square root $9x + \frac{1}{x} = \pm 10$ Hence . . . . $x = 1$ or -1.

(**15.**) The first member of (**15**) is the same as (**14.**) Hence, add unity to both members and extract square root; we then have $9x + \frac{1}{x} = \frac{29}{x} + 4$

$$9x^2 - 4x = 28 \qquad \text{Put } x = \frac{u}{9}.$$

$$u^2 - 4u = 28 \cdot 9 = 252$$

$$u - 2 = \pm 16 \text{ . . . . } x = 2.$$

(Art. 105.)

(**4.**) Multiply every term by x, and

$$x^4 - 8x^3 + 19x^2 - 12x = 0$$

Operate for square root thus

$$\begin{array}{rl}
 & x^4 - 8x^3 + 19x^2 - 12x \ (x^2 - 4x \\
 & x^4 \\
\hline
2x^2 - 4x) & -8x^3 + 19x^2 \\
 & -8x^3 + 16x^2 \\
\hline
 & \qquad 3x^2 - 12x \\
 & \qquad 3(x^2 - 4x)
\end{array}$$

$$(x^2 - 4x)^2 + 3(x^2 - 4x) = 0$$

Divide by $(x^2 - 4x)$ and

$$x^2 - 4x + 3 = 0 \text{ . . . . } x = 1 \text{ or } 3$$

But the factor $x^2 - 4x$ gives $x = 0$ or 4.

(**5.**) $x^4-10x^3+35x^2-50x+24=0$ (x^2-5x

$$x^4$$

$$2x^2-5x)-10x^3+35x^2$$

$$-10x^3+25x^2$$

$$10x^2-50x+24$$

$$(x^2-5x)^2+10(x^2-5x)+24=0$$

If we add unity to both members, we shall have complete squares. Extract square root, and

$$(x^2-5x)+5=\pm 1$$

$$x^2-5x=-4 \text{ or } -6$$

From these two equations we find $x=1$, 2, 3, or 4.

(**6.**) By mere inspection we perceive that this equation can take the form $(x^2-x)^2-(x^2-x)=132$.

$$y^2-y=132 \ldots\ldots\ldots\ldots y=12 \text{ or } -11.$$

$$x^2-x=12 \text{ or } -11$$

If we take -11, the value of x will become imaginary, 12 gives $x=4$ or -3.

(*U.* 169.)

(**7.**) This equation may be put into this form:

$$(y^2-cy)^2-2(y^2-cy)=c^2$$

from which the reduction is easy.

(Art 107.)

(**3.**) Taken from the work we have

$$(a+1)x^2-a^2x=a^2$$

or $\ldots\ldots\ldots (a+1)x^2=(x+1)a^2$

Both members are of exactly the same form, and of course the equation could not be verified unless $\cdot\cdot\, x=a$.

EXAMPLES.

(**1.**) $x^2+11x=80$. Multiply by 4, &c.

$$4x^2+\mathcal{A}+11^2=320+121=441$$

$$2x+11=\pm 21 \ldots\ldots\ldots\ldots x=5 \text{ or } -16.$$

(2.) Drop $2x$ from both members, and divide by 3; then

$$x-\frac{x-1}{x-3}=\frac{x-2}{2}$$

Clearing of fractions and

$$2x^2-6x-2x+2=x^2-3x-2x+6$$

$$x^2-3x=4 \qquad\qquad \text{Put } \ 2a=3$$

Hence, (Art. 106) . . . . $x=4$ or -1

(3.) Multiply the equation by $6x$; then

$$\frac{6x^2}{x+1}+6x+6=13x$$

$$\frac{6x^2}{x+1}+6=7x$$

$$6x^2+6x+6=7x^2+7x$$

Hence $x^2+x=6$ . . . . $x=2$ or -3.

(4.) Clearing of fractions

$$70x-21x^2+72x=500-150x$$

$$21x^2-292x=-500$$

(5.) Put $\left(\frac{6}{y}+y\right)=x$. Then

$x^2+x=30$ or, ... $x=5$ or -6.

Now, ... $\frac{6}{y}+y=5$ or -6

$$y^2-5y=-6 \quad \text{or} \quad y^2+6y=-6$$

$$4y^2-A+25=25-24$$

$2y-5=\pm 1$ . . . . $y=3$ or 2.

(6.) Put $x^{\frac{2}{3}}=y$; then $y^2+7y=44$

$$4y^2+A+49=225$$

$2y+7=\pm 15$ . . . . . . $y=4$ or -11.

$$x=(4)^{\frac{3}{2}} \text{ or } (-11)^{\frac{3}{2}}.$$

(7.) $x^2+x=42$. Hence $x=6$ or -7.
That is $y^2+11=36 \quad 49$ or $\cdots\cdots y=5$ or $\sqrt{38}$.

(8.)
$$11-\frac{x+7}{x-7}=\frac{x}{3}$$
$$33x-231-3x-21=x^2-7x$$
$$x^2-37x=-252$$
$$4x^2-A+37^2=1369-1008=361$$
$$2x-37=\pm19 \cdots\cdots\cdots\cdots x=28 \text{ or } 9.$$

(9.)
$$3x^2-9x=84$$
$$12$$
$$36x^2-A+81=12\cdot84+81=1089$$
$$6x-9=\pm33.$$

(10.) Clearing of fractions we have
$$2x+2\sqrt{x}=16-x$$
$$3x+2\sqrt{x}=16$$
Multiply by 12, &c.
$$6\sqrt{x}+2=\pm14\cdots\cdots x=2 \text{ or } 7\tfrac{1}{9}.$$

(11.)
$$\frac{6(2x-11)}{x-3}+4x=26$$
$$\frac{3(2x-11)}{x-3}+2x=13$$
$$6x-33+2x^2-6x=13x-39$$
$$2x^2-13x=-6$$
$$16x^2-A+13^2=169-48=121$$
$$4x-13=\pm11.$$

(12.) Multiply by x^3 and we have
$$10x-14+2x=\frac{22x^2}{9}$$
$$6x-7=\frac{11x^2}{9}$$
$$11x^2-54x=-63$$

Put $x=\frac{u}{11}$; then $u^2-54u=-693$

$$u^2-54u+27^2=36$$

$$u-27=\pm6 \ldots\ldots\ldots u=33 \text{ or } 21.$$

(**13.**) Clearing of fractions we have

$$x^3-10x^2+1=x^3-6x^2+9x-3x^2+18x-27$$

$$-x^2=27x-28$$

$$x^2+27x=28 \qquad \text{Put } 2a=27$$

$$x^2+2ax=2a+1$$

$$x^2+2ax+a^2=a^2+2a+1$$

$$x+a=\pm(a+1) \ \ldots x=1 \text{ or } -28$$

(**14.**) Given $mx^2-2mx\sqrt{n}=nx^2-mn$ to find x

By transposition $mx^2-2mx\sqrt{n}+mn=nx^2$

Square root . . $\sqrt{m}x-\sqrt{mn}=\pm\sqrt{n}x$

By transposition $(\sqrt{m}\pm\sqrt{n})x=\sqrt{mn}$

$$x=\frac{\sqrt{mn}}{\sqrt{m}\mp\sqrt{n}}$$

CHAPTER II.

QUADRATIC EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

Problems **1**, **2**, and **3**, require no aid from a key

(**4.**) Put $x=vy$; then the equations become

$$v^3y^3+y^3=18vy^2 \ldots\ldots\ldots (1)$$

$$vy+y=12 \ldots\ldots\ldots (2)$$

From (1) we have . . . . $y=\frac{18v}{v^3+1}$

From (2) we have . . . . $y=\frac{12}{v+1}$

Hence . . . . $\frac{2}{v+1}=\frac{3v}{v^3+1}$

Divide the denominators by $(v+1)$, and $2=\frac{3v}{v^2-v+1}$

Or, $2v^2-5v=-2$ . . . . $v=2$ or $\frac{1}{2}$.

Another Solution.

Cube the 2d equation and we have

$$x^3+y^3+3xy(x+y)=12^3$$

That is . . . . $18xy+3xy(12)=12\cdot12\cdot12$

Divide by 6, and we have $3xy+6xy=2\cdot12\cdot12$

Or . . . . . . $9xy=2\cdot3\cdot4\cdot3\cdot4$

Or . . . . . . $xy=2\cdot4\cdot4=32$

Now, having $(x+y)$ and (xy), the rest is obvious

(**5.**) The first equation can be put in this form

$$(x+y)^2+2(x+y)=120$$

Or . . . . . . $s^2+2s+1=121$

$$s+1=\pm11$$

$$x+y=10 \text{ or } -12$$

Or . . . . . . $x=10-y$ or $x=-y-12$

From the second equation $x=\frac{8+y^2}{y}$

Hence . . $\frac{8+y^2}{y}=10-y$ or, $\frac{8+y^2}{y}=-y-12$

$$8+2y^2=10y \quad \text{or,} \quad 8+2y^2+12y=0$$

$$y^2-5y=-4 \qquad y^2+6y=-4$$

$$y^2+2ay=2a+1$$

$$y+a=\pm(a+1)$$

(6.) Put $x=vy$; then the equations become

$$v^2y^2+vy^2=56$$
$$vy^2+2y^2=60$$
$$y^2=\frac{56}{v^2+v}$$
$$y^2=\frac{60}{v+2}$$

Hence .... $\frac{15}{v+2}=\frac{14}{v^2+v}$

$$15v^2+15v=14v+28$$
$$15v^2+v=28$$
$$4\cdot15^2v^2+\mathcal{A}+1=28\cdot60+1=1681$$
$$2\cdot15v+1=\pm41 \ldots\ldots v=\tfrac{4}{3} \text{ or } -\tfrac{7}{5}.$$

(7.) Put $x=vy$; then the equations become

$$6v^2y^2+2y^2=5vy^2+12 \ldots\ldots (1)$$
$$2vy^2+3v^2y^2=3y^2-3 \ldots\ldots\ldots (2)$$

From (1), ... $y^2=\frac{12}{6v^2-5v+2}$

From (2), ... $y^2=\frac{-3}{3v^2+2v-3}$

Hence ... $\frac{4}{6v^2-5v+2}+\frac{1}{3v^2+2v-3}=0$

$$12v^2+8v-12+6v^2-5v+2=0$$
$$18v^2+3v-10=0$$
$$9v^2+\tfrac{3}{2}v=5$$

Complete the square, (Art. 99.) $9v^2+\frac{3}{2}v+t^2=t^2+5$

$$3vt=\tfrac{3}{4}v \quad \text{or,} \ldots\ldots\ldots t=\tfrac{1}{4} \quad t^2=\tfrac{1}{16}$$
$$9v^2+\tfrac{3}{2}v+\tfrac{1}{16}=\tfrac{1}{16}+5=\tfrac{81}{16}$$
$$3v+\tfrac{1}{4}=\pm\tfrac{9}{4} \ldots\ldots\ldots\ldots v=\tfrac{2}{3} \text{ or } -\tfrac{5}{6}.$$

(8.) Put $x=vy$; then the equations become

$$3v^2y^2+vy^2=68 \ldots\ldots\ldots\ldots (1)$$
$$4y^2+3vy^2=160 \ldots\ldots\ldots\ldots (2)$$

$$y^2=\frac{68}{3v^2+v} \qquad y^2=\frac{160}{4+3v}$$

$$\frac{40}{4+3v}=\frac{17}{3v^2+v}$$

$$120v^2+40v=51v+68$$

$$120v^2-11v=68$$

$$4\cdot(120)^2v^2-A+121=68\cdot480+121=32761$$

$$2\cdot120v-11=\pm181 \;\ldots\ldots\ldots\; v=\tfrac{4}{5} \text{ or } -\tfrac{7}{24}.$$

(Art. 111.)

(**2.**) From the second equation we get

$$x^{\frac{1}{3}}y^{\frac{1}{3}}=6 \qquad \text{or,} \qquad 2x^{\frac{1}{3}}y^{\frac{1}{3}}=12$$

This added to the first equation gives

$$x^{\frac{1}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{1}{3}}+2(x^{\frac{1}{3}}+y^{\frac{1}{3}})=35$$

Put $x^{\frac{1}{3}}+y^{\frac{1}{3}}=s$; then $s^2+2s=35\ldots s=5$ or -7.

Now if we put $x^{\frac{1}{3}}=P$, and $y^{\frac{1}{3}}=Q$, we shall have

$$P+Q=5 \quad \text{and} \quad PQ=6$$

From which P and Q are readily found, and from them x and y.

(**3.**) Put $x^{\frac{1}{3}}=P$, and $y^{\frac{1}{3}}=Q$. Then the equations become

$$P+Q=8 \ldots\ldots\ldots\ldots (1)$$

$$P^2+Q^2+P^2Q^2=259 \ldots\ldots (2)$$

Square (1) and we have $P^2+2PQ+Q^2=64\ldots(3)$

Subtract (3) from (2), and we have $P^2Q^2-2PQ=195$

Hence $\ldots\ldots\ldots\ldots\ldots\ldots$ $PQ=15$ or -13.

Now as we have $P+Q=8$, and $PQ=15$, we have $P=5$ or 3, and $Q=3$ or 5. That is, $x^{\frac{1}{3}}=5$ or 3, &c.

(**4.**) Put $x^{\frac{2}{3}}=P$, and $y^{\frac{1}{5}}=Q$; then the equations become $P^2+Q^2+P+Q=26$, and $PQ=8$

$$\begin{array}{l} 2PQ \qquad\qquad =16 \\ \hline (P+Q)^2+(P+Q)=42 \end{array} \quad \text{Hence} \ldots P+Q=6.$$

(5.) Put $\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}=u$; then $u^2+4u=\frac{33}{4}\ldots u=\frac{3}{2}$ or $-\frac{11}{2}$.

The remaining operation is obvious.

(6.) Given $y^2-8x^{\frac{1}{2}}y=64$, and $y-2x^{\frac{1}{2}}y^{\frac{1}{2}}=4$, to find x and y.

To both members of the first equation add $16x$, and to the second add x, to complete the squares; then extract square root, and we have

$$y-4x^{\frac{1}{2}}=4(x+4)^{\frac{1}{2}} \quad \text{and} \quad y^{\frac{1}{2}}-x^{\frac{1}{2}}=(x+4)^{\frac{1}{2}}$$

Four times the last equation subtracted from the preceding gives $y-4y^{\frac{1}{2}}=0$ or $\ldots\ldots\ldots\ldots y=16$.

(7.) Multiply in the first equation as indicated, and subtract the second equation; we then have

$$x+y+2x^{\frac{1}{2}}y^{\frac{1}{2}}=25 \quad \text{or} \quad x^{\frac{1}{2}}+y^{\frac{1}{2}}=5$$

But from the second equation we have

$$(x^{\frac{1}{2}}+y^{\frac{1}{2}})x^{\frac{1}{2}}y^{\frac{1}{2}}=30 \quad \text{Hence} \ldots\ldots x^{\frac{1}{2}}y^{\frac{1}{2}}=6.$$

(8.) Divide the first equation by $y^{\frac{3}{2}}$, and $x^{\frac{2}{3}}=2y^{\frac{1}{2}}$, or $y^{\frac{1}{2}}=\frac{1}{2}x^{\frac{2}{3}}$ This put in the second equation gives

$$8x^{\frac{1}{3}}-\tfrac{1}{2}x^{\frac{2}{3}}=14$$

$$x^{\frac{2}{3}}-16x^{\frac{1}{3}}+64=64-28=36$$

CHAPTER III.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

We pass to the sixth.

(6.) Let $t=$ the time (hours) he traveled, and $r=$ his rate per hour; then $rt=36\ldots\ldots\ldots\ldots\ldots(1)$

But if r becomes $(r+1)$, t must become $(t-3)$, and then

$$(r+1)(t-3)=36 \quad \ldots\ldots\ldots\ldots (2)$$

Or . . . . . $rt+t-3r-3=36$

$$\begin{array}{l} rt+t-3r-3=36 \\ rt \qquad\qquad =36 \\ \hline t=3(r+1) \end{array}$$

Hence . . . . $r^2+r=12$, and . . . . $r=3$.

(7.) Let $x=$ the number of children,

And . . . $y=$ the original share of each.

Then . . . $xy=46800 \ldots\ldots\ldots\ldots\ldots (1)$

$$(x-2)(y+1950)=46800 \ldots\ldots\ldots (2)$$

$$xy+1950x-2y-2\cdot1950=46800$$

$$1950(x-2)=2y$$

Or . . . . . $975(x-2)x=xy=46809$

By division, $x^2-2x=48 \ldots\ldots\ldots\ldots\ldots x=8$.

(8.) Let $x=$ the number of pieces.

Then . . . . . . $\frac{675}{x}=$ the cost of each piece.

$$48x-\frac{675}{x}=675$$

$$48x^2-675x=675$$

$$16x^2-225x=225$$

(9.) Let $x=$the purchase money.

Then $\frac{104x}{100}=$ the cost, and $390-\frac{104x}{100}=$ his whole gain.

Then $\frac{104x}{100} : 390-\frac{104x}{100} :: 100 : \frac{x}{12}$

Product of extremes and means

$$\frac{26x^2}{300}=39000-104x$$

$$\frac{2x^2}{300}=3000-8x$$

Put $a=300$ and divide by 2; then

$$\frac{x^2}{a}=5a-4x$$

$$x^2+4ax=5a^2$$

$$x^2+4ax+4a^2=9a^2$$

$$x+2a=3a \ldots\ldots\ldots\ldots x=a=300.$$

(**10.**) Put $x+y=$ the greater part,
And . . . $x-y=$ the less part.
Then $2x=60$, $x=30$, and $x^2-y^2=704$.

(**11.**) Let $x=$ the cost; then $39-x=$ the whole gain.

$$x \ : \ 39-x \ : : \ 100 \ : \ x \qquad \textit{Ans. } x=10.$$

(**12.**) Let $(x-20)=$ the persons relieved by B.
Then . . . $x+20$ = the persons . . . . A.

$$\frac{1200}{x+20}+5=\frac{1200}{x-20}$$

Divide by 5, and put $a=240$; then

$$\frac{a}{x+29}+1=\frac{a}{x-20}$$

$$ax-20a+x^2-400=ax+20a$$

$$x^2=40a+400=40(a+10)=40 \cdot 250$$

Or . . . . . $x^2=400 \cdot 25 \ldots\ldots\ldots x=20 \cdot 5=100.$
Hence 80 is B's number, and 120 A's.

(**13.**) Let $x=$ the price of a dozen sherry,
And $y=$ the price of a dozen claret.

$$7x+12y=50 \ldots\ldots\ldots\ldots\ldots (1)$$

$\frac{10}{x}=$ the number of dozen of sherry for £10

$\frac{6}{y}=$ the number of dozen claret for £6.

Then . . . $\frac{10}{x}=3+\frac{6}{y} \ldots\ldots\ldots\ldots\ldots\ldots (2)$

Or . . . . . $x=\dfrac{10}{3+\frac{6}{y}}=\dfrac{10y}{3y+6}$

By substitution, $\dfrac{70y}{3y+6}+12y=50$

$$70y+36y^2+72y=150y+50\cdot 6$$

$$36y^2-8y=300$$

$9y^2-2y=75$ Hence . . . . $y=3$.

(**14.**) Let $19x=$ the whole journey.

Then $x=$ B's days, also his rate per day.

Or $x^2=$ B's distance.

Also, $7x+32=$ A's distance.

$$x^2+7x+32=19x$$

$$x^2-12x=-32$$

Hence . . . . $x=8$ or 4.

And . . . . $19x=152$ or 76.

If we put x for the whole journey, we shall obtain the 13th equation, (Art. 104.)

(**15.**) Let $x=$ the bushels of wheat,

And . . . $x+16=$ the bushels of barley.

$$\frac{24}{x}=\frac{24}{x+16}+\frac{1}{4}$$

$$24x+16\cdot 24=24x+\frac{x^2+16x}{4}.$$

$$x^2+16x=16\cdot 96=16\cdot 16\cdot 6$$

Put $2a=16$. Then $2a\cdot 2a\cdot 6=24a^2$

$$x^2+2ax=24a^2$$

$x+a=\pm 5a$ . . . . $x=4a=32$.

(**16.**) A put in 4 horses, and B put in x horses.

Then $\dfrac{18}{x}=$the rate per head.

$$\frac{4 \cdot 18}{x}+18= \text{ the price of the pasture.}$$

$$\frac{4 \cdot 20}{x+2}+20= \text{ the price of the pasture.}$$

Hence . . . . $\dfrac{4 \cdot 18}{x}=\dfrac{4 \cdot 20}{x+2}+2$

$\dfrac{36}{x}=\dfrac{40}{x+2}+1$ . . . . $x=6$

(**17.**) Let $4x=$ the price per yard,
And . . . $9x=$ the number of yards.
$36x^2=324$ . . . . $x=3$.

(**18.**) Let $10x+y=$ the number.

Then . . . . . . $\dfrac{10x+y}{xy}=2$ . . . . (1)

And . . . . . . $10x+y+27=10y+x$ . . . . (2)

From (1) . . . . $10x=(2x-1)y$

From (2) . . . . $x+3=y$

By division, . . $\dfrac{10x}{x+3}=2x-1$

$10x=2x^2+6x-x-3$

$2x^2-5x=3$ . . . . $x=3$.

(**19.**) Let $(x-y)$, x, and $(x+y-6)$ represent the numbers. Then $3x-6=33$, or $x=13$.

$$(x-y)^2=x^2-2xy+y^2$$
$$x^2=x^2$$
$$(x+y-6)^2=x^2+2xy+y^2-12x-12y+36$$
$$\overline{3x^2+2y^2-12x-12y+36=441}$$

By subtracting the value of $3x^2-12x+36$, we have
$2y^2-12y=54$ Hence . . . . . . $y=9$.

(**20.**) Let x and y be the numbers.

Then . . . . $x+y=xy$ . . . . (1)

$x+y=x^2+y^2$ . . . . (2)

Put $x=vy$, and the equations become

$vy+y=vy^2$ . . . . (3)

$vy+y=v^2y^2+y^2$ . . . . (4)

Divide (4) by (3), and $\dfrac{v^2+1}{v}=1$

This gives $v=\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$

But from equation (3) we have $vy=v+1=x=\frac{3}{2}\pm\frac{1}{2}\sqrt{-3}$

(**21.**) Let $x+y=$ the greater number,

And . . . . $x-y=$ the less.

$x^2-y^2=24$ . . . . (1)

$2x+2x^2+2y^2=62$ . . . . (2)

Or . . . . $x+x^2+y^2=31$

Add (1) . . . . . $x^2-y^2=24$

$2x^2+x=55$ $\quad x=5, \quad y=1.$

(**22.**) Let $x+y=$ the greater number,

And . . . $x-y=$ the less.

$x^2-y^2+2x=47$ . . . . (1)

$2x^2+2y^2-2x=62$ . . . . (2)

$2x^2-2y^2+4x=94$

$4x^2+2x=156$

$2x^2+x=78$ $\quad x=6, \quad y=1.$

(**23.**) Let $(x+y)=$ one number,

And . . . . $(x-y)=$ the other.

$2x=27$ . . . . (1)

$x^3+3x^2y+3xy^2+y^3$

$x^3-3x^2y+3xy^2-y^3$

$2x^3+6xy^2=5103$ . . . . (2)

Divide (2) by (1), and we have $x^2+3y^2=189$

$$\frac{3^2 \cdot 9^2}{4}+3y^2=9\cdot21$$

Divide by 3, and $\frac{3\cdot81}{4}+y^2=3\cdot21$

$$3\cdot81+4y^2=3\cdot84$$

$$4y^2=3\cdot3 \quad \text{or} \quad 2y=3 \quad y=\tfrac{3}{2}.$$

$$x+y=\tfrac{27}{2}+\tfrac{3}{2}=15, \text{ \&c.}$$

(**24.**) Let $x+y=$ one number,

And . . . . $x-y=$ the less.

$$2x=9$$

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

$$x^4-4x^3y+6x^2y^2-4xy^3+y^4$$

$$2x^4+12x^2y^2+2y^4=2417$$

By resolving this we shall find $y=\frac{5}{2}$.

(**25.**) Let $x+y=$ the greater, and $x-y=$ the less.

Then . . . . . $(x^2-y^2)(2x^2+2y^2)=1248$ . . . . . . (1)

Or . . . . $x^4-y^4=624$

Also . . . . $4xy=20$ . . . . (2)

Or . . . . $y=\frac{5}{x}$

$$y^4=\frac{625}{x^4}$$

$$x^4-\frac{625}{x^4}=624$$

$x^8-624x^4=625$ Put $2a=624$

Then . . . . $x^8-2ax^4+a^2=a^2+2a+1$

$$x^4-a=\pm(a+1)$$

$x^4=2a+1$ or -1 . . . . $x=5$

$y=1$.

(**26.**) Let $x=$ the days required by one,

And . . . . $x+10=$ the days required by the other.

Then . . . $\frac{1}{x}$= one day's work of the first.

$\frac{1}{x+10}$= one day's work of the second.

$$\frac{1}{x}+\frac{1}{x+10}=\frac{1}{12}$$

SECTION V.

CHAPTER I.

ARITHMETICAL PROGRESSION.

(Art. 116.)

(4.) $L=1+31d$ . . . . (A)

$280=16(2+31)d$ . . . . (B)

(B) reduced gives $d=\frac{1}{2}$; then (A) gives $L=16\frac{1}{2}$.

(5.) Here $a=\frac{1}{3}$, $L=\frac{1}{2}$, $n=5$.

$\frac{1}{2}=\frac{1}{3}+4d$ . . . . (A)

$s=(\frac{1}{3}+\frac{1}{2})\frac{5}{2}$ . . . . (B)

From (A) we have $d=\frac{1}{24}$

$\frac{1}{3}+\frac{1}{24}=\frac{3}{8}=$ the first mean, &c.

(6.) Here $a=9$, $L=109$, $n=11$.

$109=9+10d$ . . . . . . $d=10$, *Ans.*

(7.) $L=1+364\times2$ . . . . (A)

$s=730\times365\times\frac{1}{2}=(B)$, *Ans.*

(8.) $L=20+(n-1)3=17+3n$ . . . . (A)

$s=(37+3n)\frac{1}{2}n=438$ . . . . (B)

(Art. 117.)

(2.) Represent the numbers by $x-y$, x, and $x+y$.

$3x=18$ $\qquad$ $x=6$

$$(x-y)^2=x^2-2xy+y^2$$
$$x^2=x^2$$
$$(x+y)^2=x^2+2xy+y^2$$
$$\overline{3x^2+2y^2=158}$$

(**3.**) Let $x-3y$, $x-y$, $x+y$, and $x+3y$ represent the numbers; then $2y=4$.

The product of the 1st and 4th is

x^2-9y^2; of the 2d and 3d is (x^2-y^2).

$$\begin{array}{l} x^2-y^2 \\ \hline x^4-\ 9x^2y^2 \\ \quad -\ \ x^2y^2+9y^4 \\ \hline x^4-10x^2y^2+9y^4=176985 \\ \qquad\qquad\qquad 9y^4=\quad 144 \\ x^4-40x^2 \qquad\qquad =176841 \end{array}$$

(**4.**) The same notation as in last example.

$2x=8$ $\qquad$ $x=4$ $\qquad$ $x^2-y^2=15$

(**5.**) Let $n=$ the number of days.

Then . . . . $L=1+(n-1)1=n$

$s=(1+n)\frac{1}{2}n=$ the whole distance.

Also . . . . $(n-6)15$ $=$ the whole distance.

$$n^2+n=30n-180$$
$$n^2-29n=-180 \quad \ldots\ldots n=9 \text{ or } 20$$
$$\begin{array}{rr} 6 & 6 \\ \hline 3 & 14 \end{array}$$

(**6.**) The first day he must pay $1+i$; i representing the interest of one dollar for one day.

First day . . . . $1+\ \ i$

2d day . . . . $1+\ 2i$

3d day . . . . $1+\ 3i$

Last day . . . . $1+60i$

$(2+61i)30=$ the whole sum

to be paid; but as this sum is to be paid in 60 equal payments, each payment must be

$$1+\frac{61i}{2}= \textit{Ans.}\ \$1 \text{ and } \tfrac{5}{6} \text{ of a cent, nearly.}$$

(7.) Let $x-3y$, $x-y$, $x+y$, and $x+3y$ represent the numbers; then

$$\begin{array}{r} 2x^2+18y^2=50 \\ 2x^2+\ 2y^2=34 \\ \hline 16y^2=16 \end{array} \cdots\cdots\cdots y=1.$$

GEOMETRICAL PROGRESSION AND HARMONICAL PROPORTION.

(Art 124.)

(**1.**) Let x represent the mean sought.

$$x=\frac{2\cdot 6\cdot 12}{18}=8$$

(**2.**) Let $x=$ the number sought. Then, by harmonical proportion

$$234 : x :: 90 : 144-x$$

$$90x=234\cdot 144-234x$$

$$324x=234\cdot 144 \quad \text{Hence} \cdots\cdots x=104.$$

(**3.**) Let $x=$ the number sought.

Then $\cdots\cdot 24 : x :: 8 : 4-x$

Or $\cdots\cdots 3 : x :: 1 : 4-x \quad \cdot\cdot x=3.$

(**4.**) Let $x=$ the second.

Then $\cdot\cdot\cdot 16 : 2 :: 16-x : 1 \cdots\cdot x=8.$

(**5.**) Let $x=$ the first number, and $y=$ ratio.

Then $\cdots\cdots\cdot x+xy+xy^2=210 \cdots\cdots\cdots (1)$

$$xy^2-x=\ 90 \cdots\cdots\cdots (2)$$

By subtraction $\cdots\cdot 2x+xy=120$ or $x=\dfrac{120}{2+y}$

From (2) we have $\cdots\cdots\cdots\cdots\cdots x=\dfrac{90}{y^2-1}$

$$\frac{4}{2+y}=\frac{3}{y^2-1} \quad \text{or,} \quad 4y^2-3y=10 \,.\,.\,.\,.\,.\, y=2.$$

(5.) Let x, xy, xy^2, and xy^3 represent the numbers.

Then . . . $\dfrac{xy^3}{xy+xy^2}=\dfrac{y^2}{1+y}=\dfrac{4}{3}$.

From this equation we perceive at once that $y=2$; then

$$x+2x+4x+8x=15x=30 \,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\, x=2.$$

(6.) Let x, xy, xy^2 and xy^3 represent the numbers.

$x+xy^2=148$ . . . . (1)

$xy+xy^3=888$ . . . . (2)

Or . . . . $x(1+y^2)=4\cdot37$ . . . . (3)

$xy(1+y^2)=4\cdot222$ . . . . (4)

Divide (4) by (3), and . . . . $y=6$.

(7.) Let x, $\sqrt{xy}$, and y represent the numbers; then

$x+\sqrt{xy}+y=14$ . . . . (1)

And . . . . . . $x^2+xy+y^2=84$ . . . . (2)

Put $x+y=s$, and $\sqrt{xy}=p$;

Then . . $x^2+xy+y^2=s^2-p^2$, and equations (1) and (2) become . . . . $s+p=14$ . . . . (3)

$s^2-p^2=84$ . . . . (4)

Divide (4) by (3), and we have $s-p=6$ . . . . . (5)

Add (3) to (5), and divide by 2, and

$s=10$. Hence $p=4$.

(8.) Let x, xy, xy^2, and xy^3 represent the numbers;

Then . . . . $xy^3-xy=24$

$$xy^3+x \; : \; xy^2+xy \; :: \; 7 \; : \; 3$$

Or . . . $y^3+1 \; : \; y^2+y \; :: \; 7 \; : \; 3$

Divide the first couplet by $(y+1)$, and we have

$$y^2-y+1 \; : \; y \; :: \; 7 \; : \; 3$$

$$3y^2-3y+3=7y \quad \text{or} \quad 3y^2-10y=-3$$

From this equation we have $y=3$, the ratio.

(**9.**) Let x, xy, xy^2, and xy^3 represent the numbers;

Then . . . . . . $x(1+y+y^2+y^3)=y+1$

And . . . . . . $x=\frac{1}{10}$. Put $(y+1)=A$.

Then . . . . . . $\frac{1}{10}(A+Ay^2)=A$

$A+Ay^2=10A \qquad Ay^2=9A$ or . . . . . . $y=3$.

Hence $\frac{1}{10}$, $\frac{3}{10}$, &c. are the numbers.

(**10.**) Let x, $\frac{2xy}{x+y}$, and y represent the numbers; then

$$x+\frac{2xy}{x+y}+y=26 \text{ . . . . } (1)$$

And . . . . $xy=72$

Put $x+y=s$; then equation (1) becomes

$$s+\frac{144}{s}=26 \quad \text{or} \quad s^2-26s=-144 \text{ . . . . } s=18.$$

(**11.**) Let x, xy, and xy^2 represent the numbers;

Then . . . . $x^3y^3=216$ . . . . (1)

$$x^2+x^2y^4=328 \text{ . . . . } (2)$$

From (1) . . . . . $xy=6$, or $x^2=\frac{36}{y^2}$

From (2) . . . . $x^2=\frac{328}{1+y^4}$

$$\frac{36}{y^2}=\frac{328}{1+y^4} \quad \text{or} \quad \frac{9}{y^2}=\frac{82}{1+y^4}$$

$9y^4-82y^2=-9$ Hence . . . . . . $y=3$.

(**12.**) Let x, $\sqrt{xy}$, and y represent the numbers; then

$$x+\sqrt{xy}+y=13 \text{ . . . . } (1)$$

$$(x+y)\sqrt{xy}=30 \text{ . . . . } (2)$$

$$x+y=13-\sqrt{xy} \text{ . . . . } (3)$$

$$x+y=\frac{30}{\sqrt{xy}} \text{ . . . . } (4)$$

$$13-\sqrt{xy}=\frac{30}{\sqrt{xy}} \quad \text{Hence . . } \sqrt{xy}=3$$

(**13.**) Let x, $\frac{2xy}{x+y}$, and y represent the numbers; then

$$x+y=18 \dots\dots (1)$$

$$\frac{2x^2y^2}{18}=576 \dots\dots (2)$$

$$\frac{xy}{3}=24 \qquad xy=72 \dots\dots (3)$$

From (1) and (3) we find x and y.

(**14.**) Let x, xy, and xy^2 represent the numbers; then (xy^2-xy) $(xy-x)$ are the 1st differences, and

$$xy^2-2xy+x=6$$

$$xy^2+xy+x=42$$

Difference $\dots\dots$ $3xy=36 \dots\dots xy=12.$

(**15.**) Let x, $\frac{2xy}{x+y}$, and y represent the numbers. If y is supposed greater than x, then $\left(y-\frac{2xy}{x+y}\right)\left(\frac{2xy}{x+y}-x\right)$ are the 1st differences, and $y-\frac{4xy}{x+y}+x=2$ the 2d diff.

$xy=72$ Put $(x+y)=s$;

Then $\dots\dots$ $s-\frac{4\cdot72}{s}=2$

$$s^2-2s+1=289$$

$$s-1=17 \dots\dots s=x+y=18.$$

(**17.**) Let x^2, xy, and y^2 represent the numbers; then

$$x^2+xy+y^2=31, \quad \text{and} \quad x^2+y^2=26$$

(**18.**) Let x, xy, xy^2, xy^3, xy^4, and xy^5 represent the numbers. Then, by the conditions, we have

$$x+xy+xy^2+xy^3+xy^4+xy^5=189=a \dots (1)$$

And $xy+xy^4=54=b \dots\dots (2)$

But equation (1) may be put into this form

$$(1+y+y^2)x+(1+y+y^2)xy^3=a$$

Or . . . . . $x+xy^3=\dfrac{a}{1+y+y^2}$

Multiply this last equation by y, and its first member will be the same as the first member of equation (2); therefore $\dfrac{ay}{1+y+y^2}=b$; a quadratic from which we obtain y, the ratio.

(**19.**) Take the same notation as for (**18**); then we have

$$(x+xy)+(x+xy)y^4=189-36=153=a \;\cdot\cdot\; (1)$$

And . . . . $(x+xy)y^2=36=b$ . . . . (2)

Divide (1) by (2), and we have

$$\frac{1+y^4}{y^2}=\frac{153}{36}=\frac{51}{12} \quad \text{Hence} \;\cdots\; y=2.$$

CHAPTER III.

PROPORTION.

(**5.**) Let x and y represent the numbers; then

$$x-y : x+y :: 2 : 9$$
$$x+y : xy :: 18 : 77$$

From the first, $2x : 2y :: 11 : 7$ or, $x=\frac{11}{7}y$.

$$\frac{18y}{7} : \frac{11y^2}{7} :: 18 : 77$$
$$y : 11y^2 :: 1 : 77 \qquad y=7.$$

(**6.**) Let x and y represent the numbers.

$$x+4 : y+4 :: 3 : 4 \;\cdots\; (1)$$
$$x-4 : y-4 :: 1 : 4 \;\cdots\; (2)$$

From (2) we have $4x-16=y-4$, or, $y=4x-12$. This value of y put in (1) gives

$$x+4 : 4x-8 :: 3 : 4$$
$$x+4 : x-2 :: 3 : 1$$
$$x+4=3x-6 \ldots\ldots\ldots\ldots x=5.$$

(7.) Let x and y represent the numbers;

Then .... $x+y=16$

And ..... $xy : x^2+y^2 :: 15 : 34$

Double the first and third terms, then add and subtract (Theorem 4), and $2xy : x^2+y^2 :: 30 : 34$

$$x^2+2xy+y^2 : x^2-2xy+y^2 :: 64 : 4$$
$$x+y : x-y :: 8 : 2$$
$$16 : x-y :: 4 : 1 \text{ or, } x-y=4.$$

(8.) Let $x=$ the gallons of rum,

And ... $y=$ the gallons of brandy.

$$x-y : y :: 100 : x$$
$$x-y : x :: 4 : y$$

Product $(x-y)^2 : xy :: 400 : xy$

Dividing the second and fourth by xy, and

$$(x-y)^2 : 1 :: 400 : 1$$
$$x-y : 1 :: 20 : 1 \text{ or, } x-y=20.$$

(9.) Let $x+y=$ the greater number,

And ... $x-y=$ the less.

Then $x^2-y^2=320 \ldots\ldots\ldots\ldots\ldots\ldots (1)$

$$(x+y)^3=x^3+3x^2y+3xy^2+y^3$$
$$(x-y)^3=x^3-3x^2y+3xy^2-y^3$$
$$6x^2y+2y^3=\text{diff. of the cubes.}$$

$2y=$ difference. Cube of $(2y)=8y^3$

$$6x^2y+2y^3 : 8y^3 :: 61 : 1$$
$$3x^2+y^2 : 4y^2 :: 61 : 1$$
$$3x^2+y^2=244y^2 \qquad 3x^2=243y^2$$
$$x^2=81y^2$$

This value of x^2 put in equation (1), gives

$$80y^2=320 \text{ or} \ldots\ldots\ldots\ldots\ldots\ldots y=2.$$

(**10.**) Let x and y represent the numbers;

Then . . . . $x+y=60$

And . . . . $xy \ : \ x^2+y^2 \ : : \ 2 \ : \ 5$

Double the first and third terms, aud

$$2xy \ : \ x^2+y^2 \ : : \ 4 \ : \ 5$$

Add and subtract and

$$x^2+2xy+y^2 \ : \ x^2-2xy+y^2 \ : : \ 9 \ : \ 1$$

$$60 \ : \ x-y \ : : \ 3 \ : \ 1 \qquad x-y=20$$

But . . . . $x+y=60$

(**11.**) Let $3x$ and $2x$ represent the numbers;

$$3x+6 \ : \ 2x-6 \ : : \ 3 \ : \ 1 \qquad \text{Hence, } x=8$$

(**12.**) Let $16x$ and $9x$ represent the numbers;

Then . . $16x \ : \ 24 \ : : \ 24 \ : \ 9x$ Hence, $x=2$

(**13.**) Let x and y represent the numbers;

Then $x+y \ : \ x-y \ : : \ 4 \ : \ 1$ Hence, $5y=3x$

This reduced equation will be true if we take $y=3$ and $x=5$, or, if we take any multiples of 5 and 3, as 10 and 6, 15 and 9, 20 and 12, &c. Hence the answer in the book is correct, but many other answers are equally correct, and the problem is therefore indeterminate.

Again let x and y represent the numbers;

And $x^2+y^2 \ : \ x \ : : \ 102 \ : \ 5$

Or, $x^2+\frac{9}{25}x^2 \ : \ x \ : : \ 102 \ : \ 5$ Hence $x=15$

(**14.**) Let x and y rcpresent the two numbers

Then, $x+y=20$, and $x \ : \ y \ : : \ 9 \ : \ 1$

APPLICATION OF THE BINOMIAL THEOREM.

(3.) To expand $(a-b)^{-1}$, we take formula (3). Then $a=a$, $x=-b$, and $m=-1$.

Hence,

$$a^m+ma^{m-1}b+m\tfrac{m-1}{2}a^{m-2}b^2, \text{ \&c.}=a^{-1}+a^{-2}b+a^{-3}b^2, \text{ \&c.}$$

Or .... $\dfrac{1}{a}+\dfrac{b}{a^2}+\dfrac{b^2}{a^3}$, &c.

(4.) Take formula (1), and put $x=\dfrac{b^2}{a^2}$. Then

$$a(1+x)^{\frac{1}{2}}=a\left(1+\tfrac{1}{2}x+\frac{1}{2}\frac{(\frac{1}{2}-1)}{2}x^2+\frac{1}{2}\frac{(\frac{1}{2}-1)(\frac{1}{2}-2)}{2\quad 3}x^3 \text{ \&c.}\right)$$

$$a\left(1+\frac{b^2}{2a^2}-\frac{b^4}{2\cdot 4a^4}+\frac{3b^6}{2\cdot 46a^6} \text{ \&c.}\right)$$

(5.) Expand $\dfrac{d}{(c^2+x^2)^{\frac{1}{2}}}=\dfrac{d}{c\left(1+\dfrac{x^2}{c^2}\right)^{\frac{1}{2}}}=\dfrac{d}{c}\left(1+\dfrac{x^2}{c^2}\right)^{-\frac{1}{2}}$

The numerical coefficients of this series must be the same as the last, because m is numerically the same; the sign of the second and every alternate term will be minus.

$$\frac{d}{c}\left(1-\frac{x^2}{2c^2}+\frac{3x^4}{2\cdot 4c^4}-\frac{3\cdot 5x^6}{2\cdot 4\cdot 6c^6}+\frac{3\cdot 5\cdot 7x^8}{2\cdot 4\cdot 6\cdot 8c^8} \text{ \&c.}\right)$$

(6.) $(a^2)^{\frac{3}{4}}\left(1-\dfrac{x^2}{a^2}\right)^{\frac{3}{4}}=a^{\frac{3}{2}}\left(1-\dfrac{x^2}{a^2}\right)^{\frac{3}{4}}=\dfrac{a^2}{\sqrt{a}}\left(1-\dfrac{x^2}{a^2}\right)^{\frac{3}{4}}$

Here .... $m=\dfrac{3}{4}$ $\dfrac{x}{a}=\dfrac{x^2}{a^2}$

Formula (2) $1+m\dfrac{x}{a}+m\cdot\dfrac{m-1}{2}\;\dfrac{x^2}{a^2}$ &c. $=$

$\left(1-\dfrac{3}{4}\,\dfrac{x^2}{a^2}-\dfrac{3}{4\cdot 8}\,\dfrac{x^4}{a^4}-\dfrac{3\cdot 5x^6}{4\cdot 8\cdot 12c^6} \text{ \&c.}\right)$ This multiplied by the factor $\dfrac{a^2}{\sqrt{a}}$ will give $\sqrt{\dfrac{1}{a}}\left(a^2-\dfrac{3x^2}{2^2}-\dfrac{3x^4}{2^5a^2}-\dfrac{5x^6}{2^7c^4} \text{ \&c.}\right)$

(7.) $(a+y)^m+=a^m+ma^{m-1}y+m\frac{m-1}{2}a^{m-2}y^2$, &c., and if $m=-4$, the series will be

$$a^{-4}-4a^{-5}y+10a^{-6}y^2-20a^{-7}y^3+ \text{ \&c.}$$

Or . . . . . . $\frac{1}{a^4}-\frac{4y}{a^5}+\frac{10y^2}{a^6}-\frac{20y^3}{a^7}$ &c.

(8.) $\frac{a^3}{a^3+b^3}=\frac{1}{1+\frac{b^3}{a^3}}=\left(1+\frac{b^3}{a^3}\right)^{-1}$ The cube root of this

is . . . $\left(1+\frac{b^3}{a^3}\right)^{-\frac{1}{3}}=1-\frac{b^3}{3a^3}+\frac{1\cdot4b^6}{3\cdot6a^6}-\frac{1\cdot4\cdot7b^9}{3\cdot6\cdot9a^9}$ &c.

Or . . . . $=1-\frac{b^3}{3a^3}+\frac{2b^6}{9a^6}-\frac{14b^9}{81a^9}$ &c.

COMPOUND INTEREST.

(4.) The general equation is $pA^n=a$, and for this example $p=5$, $A=(1.05)$ $a=9$, and n is unknown.

Hence . . . . . . $5(1\cdot05)^n=9$, or $(1\cdot05)^n=1\cdot8$

Or . . . . $n=\frac{\log.\ 1.8}{\log.\ (1\cdot05)}=\frac{0.25527}{0.02119}=12.04$ years, nearly.

(5.) $1000(1+r)^6=1800$ or $(1+r)^6=1.8$

By logarithms . . . $6\log.(1+r)=\log.(1.8)$

Hence . . . . $\log.(1+r)=\frac{0\cdot25527}{6}=0.04254$

By the tables we find $1+r=1\cdot103$, or, $r=.10\frac{3}{10}$, nearly.

(6.) In this example the general formula is

$p(1.04)^4=350.9575$. By logarithms we have

$\log.\ p+4\log.\ (1.04)=\log.\ 350.9575$

Or . . . . $\log.\ p+0.06812=2.54525$

By reduction . . . . $\log.\ p=2.47713$

Hence . . . . $p=300$, *Ans.*

The general equation applied to this problem is

$$3600(1.05)^n=5000(1.04)^{12}$$

By reduction . . . . $(1.05)^n=\frac{50}{36}(1.04)^{12}$

$$n.\ \log.\ (1.05)=\log.\ \tfrac{50}{36}+12\ \log.\ (1.04)$$

$$n(0.02119)=0.14267+0.19236$$

$$n=\frac{0.33503}{0.02119}=16 \text{ nearly.}$$

ANNUITIES.

(6.) That is, the rent is an annuity to continue forever; what sum of money will purchase it?

The general equation is $P=\frac{p}{r}=\frac{3000}{.03}=100000.$

(7.) The general equation applied to this problem is

$$A'=\frac{350[(1.04)^8-1]}{04}=8750[(1.04)^8-1]$$

$$\text{Log. } A'=\log.\ 8750+\log.\ ((1.04)^8-1)$$

But . . . . $(1.04)^8-1=.3685$

$$\text{Log. } A'=3.94201-1.+0.56644=3.50845$$

But the answer to the question is $P=\frac{A'}{A^n}$ (Art. 155.)

That is by log. (log. P.)$=3.50845-0.13624=3.37221$

Or . . . . $P=2356.46$ *Ans.*

(8.) If no interest were required, the sum to be paid at each annual payment would be $\frac{1200}{7}$ dollars; but this must be paid and compound interest on the same for 1, 2, 3, &c. years, up to 7. Call $\frac{1200}{7}=p$.

Let $A=(1+r)=1.04$. Then by (Art. 153),

The sum to be paid the first year must be . . . . pA

. . . . second year . . . . pA^2

. . . . third year . . . . pA^3

. . . . last year . . . . pA^7

This is a geometrical series, and its sum (Art. 120) is

$$s=\frac{pA^8-pA}{A-1}=\frac{pA(A^7-1)}{r}=\frac{pA}{r}((1+r)^7-1)$$

$$\text{Or, } s=\frac{1200}{7}\times\frac{1.04}{.04}\left((1.04)^7-1\right)=\frac{300.104}{7}\times.3158$$

But, if this sum is to be paid by seven equal payments, each payment must be

$$\frac{300\times104}{49}\times.3158=\frac{312}{49}\times31.58=\$201+ \textit{ Ans.}$$

(9.) Here $P=\frac{p}{r}=\frac{250}{.07}=\$3571\frac{3}{7}$ *Ans.*

SOLUTIONS OF EQUATIONS OF THE HIGHER DEGREES.

NEWTON'S METHOD OF APPROXIMATION.

(1.) Given $x^3+2x^2-23x=70$, to find one value of x.

By trial we find that one value of x is between 5 and 6, nearer 5 than 6; therefore, let $a=5$ and $y=$ the remaining part of the root. Then $x=a+y$.

Expand, neglecting all the terms containing the powers of y after the first, and we shall have

$$\begin{aligned} x^3&=a^3+3a^2y+\text{ \&c.}\\ 2x^2&=2a^2+4ay+\text{ \&c.}\\ -23x&=-23a-23y \end{aligned}$$

By addition,

$$x^3+2x^2-23x=a^3+2a^2-23a+(3a^2+4a-23)y=70$$

In this last equation we observe that a has the same powers and coefficients as x, and the coefficients to y may be found by the following

RULE. *Multiply each coefficient of* x *by its exponent, diminish each exponent by unity, and change* x *to* a.

Now $y=\frac{70+23a-2a^2-a^3}{3a^2+4a-23}$ Giving a its value, 5, we have $y=\frac{10}{72}=.1+$ Now make $a=5.1$, and substitute again in the preceding formula, we have a new value of y.

Thus $y=\frac{2.629}{75.43}=.03$ Now make $a=5.13$, and substitute again, and our new value of y will be $.004578+$ Hence $a+y$ or $x=5.134578+$

(**2.**) Given $x^4-3x^2+75x=10000$, to find one value of x.

By trial we find x must be near 10. Hence put $a=10$ and $x=a+y$. Then by the preceding rule

$$y=\frac{10000-75a+3a^2-a^4}{4a^3-6a+75}=\frac{-450}{4015}=-.11$$

Now make $a=10-.11=9.89$. If we have the patience to substitute this value for a in the equation, we shall have a new value to y, true to 6 or 7 places of decimals, and of course a value to x to the same degree of exactness.

(**3.**) Given $3x^4-35x^3-11x^2-14x+30=0$ to find one value of x.

By trial we find that x must be near 12. Let $a=12$, and $x=a+y$. Then by the rule

$$y=\frac{-30+14a+11a^2+35a^3-3a^4}{12a^3-105a^2-22a-14}=\frac{-6}{5338}=-00112$$

Hence . . . . $x=12-.00112=11.99888.$

(**4.**) Given $5x^3-3x^2-2x=1560$, to find x.

We find by trial that one value of x is more than 7. Put $x=a+y$, and $a=7$. Then by the rule

$$y=\frac{1560+2a+3a^2-5a^3}{15a^2-6a-2}=\frac{6}{689}=.00867+$$

Hence . . . . $x=7.00867+$

CHAPTER III.

YOUNG'S METHOD OF RESOLVING THE HIGHER EQUATIONS.

This is really Horner's method, but extracted from Young.

(6.) Given $x^2+7x=1194$, to find the values of x.

We perceive by inspection that one value of x must be more than 31; therefore put $r=31$.

$a+r=$	38	1194 (31.2311+ &c.
$r+s$	31.2	1178
$a+2r+s$	69.2	1600
$s+t$	23	1384
	69.43	21600
	31	20829
	69.461	77100
	&c.	69461
		&c.

As the sum of the two roots is equal to -7, the other root must be $-38.2311+$

(7.) Not necessary to have place in a key.

(8.) Given $x^2-21x=214591760730$, to find one value of x.

In this example the pupil might be at a loss as to the most expeditious manner of finding r by trial.

Conceive $-21x$ not to exist; then the value of x will be the square root of the absolute term; but this term has six periods of two figures each, and the superior period is 21. The greatest square in this is 16, root 4; hence r must be at least 400000, six places; try this number.

$-a+r$	$=399979$	214591760730 (	$400000=r$
$r+s$	460000	1599916	$60000=s$
$-a+2r+s$	859979	5460016	$3000=t$
$s+t$ ...	63000	5159874	$200=u$
$a+2r+2s+t$.	922979	3001420	$50=v$
	3200	2768937	$1=w$
	926179	2324837	
	250	1852358	
	926429	4724793	
	51	4632145	$x=463251.$
	926480	926480	
		926480	

As the algebraic sum of the two roots must make 21, (Art. 156 in the work,) therefore the other root must be -463230.

(**9.**) Given $7x^2-3x=375$, to find one value of x.

Or $x^2-\frac{3}{7}x=\frac{375}{7}$. Put $x=\frac{1}{7}y$. (Art. 166.)

Then $\frac{y^2}{49}-\frac{3y}{49}=\frac{375}{7}$ Or $y^2-3y=375\times7=2625$.

In this equation we perceive that y must be more than the square root of 2625, that is, more than 50. Hence put $r=50$.

			$r\ s\ t$
$-a+r$	47	2625 (	52.756+
$r+s$	52	235	
$-a+2r+s$ ...	99	275	
$s+t$	2.7	198	
	1017	7700	
	75	7119	
	10245	58100	
		51225	
		6875	

Hence $x=\frac{52.756+}{7}=7.+$

(**10.**) Given $2x^2-11x=-7\frac{1}{2}$, to find one value of x.

Or $x^2-\frac{11}{2}x=-\frac{15}{4}$ Put $x=\frac{1}{2}y$.

Then . . . $\frac{1}{4}y^2-\frac{11}{4}y=-\frac{15}{4}$ or. . . $y^2-11y=-15$.

In this equation one value of y is between 9 and 10; therefore put $r=9$.

$$\begin{array}{lrr}
-a+\ r \ldots\ldots & -2 & -15\ (\ 9.405124838+ \\
\quad r+s & \underline{9.4} & \underline{-18} \\
-a+2r+s & 7.4 & 300 \\
\qquad s+t & \underline{4} & \underline{296} \\
 & 7.805 & 40000 \\
 & \underline{51} & \underline{39025} \\
 & 78101 & 97500 \\
 & \underline{12} & \underline{78101} \\
 & 781022 & 1939900 \\
 & \underline{24} & \underline{1562044} \\
 & 7810244 & 377856
\end{array}$$

The division is not carried out for the last three figures. Hence one value of y is $9.405124838+$, and as the two values must make 11, (Art. 166), the other value must be 1.594875161. But $x=\frac{1}{2}y$, therefore

$$x=.797437580+ \text{ or, } x=4.702562419+ \ \textit{Ans.}$$

(**11.**) Given $\frac{3}{4}x^2+\frac{3}{5}x=\frac{7}{11}$, to find one value of x.

Or $x^2+\frac{4}{5}x=\frac{28}{33}$ or, $x^2+.8x=0.848484848+$

Here it is obvious that x cannot be 1; by trial we find it must be near .6; therefore $r=6$.

$$\begin{array}{lrl}
a+r & 1.4 & 0.8484848484\ \&c.\ (\ 0.604233+ \\
\quad r+s & \underline{60} & \underline{84} \\
 & 2004 & 8484 \\
 & \underline{42} & \underline{8016} \\
 & 20082 & 46884 \\
 & \underline{23} & \underline{40164} \\
 & 200843 & 672084
\end{array}$$

We omit several examples as they present no difficulty.

CUBIC EQUATIONS.

(**3.**) Given $x^3+2x^2-23x=70$, to find one value of x.

By trial we find x must be a little over **5**; therefore $r=5$, $A=2$, $B=-23$, $N=70$.

B	-23	
$r(r+A)$	35	$r\ s\ t$
1st Divisor	12	70 (5.134
r^2	25	60
B'	72	10000
$s(s+3r+A)$	171	7371
2d Divisor	7371	2629000
s^2	1	2276697
B''	7543	352303000
*$(3Q+t)t$	4599	305649104
3d Divisor	758899	46653896
t^2	9	
B'''	763507	
	61576	
4th Divisor	76412276	
	16	

Common division will give three or four more figures to perfect accuracy.

(**4.**) Given $x^3-17x^2+42x=185$, to find one value of x.

Here $A=-17$, $B=42$, $N=185$, and we find by trial that x must be between 15 and 16; therefore $r=15$.

* Q represents the root as far as previously determined.

B	42	
$r(r+A)$	-30	
		$\overset{r\ s\ t}{}$
1st Divisor	12	185 (15.02
r^2	225	180
B'	207	5000000
$s(s+3r+A)$	0	4154008
2d Divisor	207	2077) 845992 (407
s^2	0	8298
B''	207	16192
$t(3Q+t)$	7004	14539
3d Divisor	2077004	1653.0

Hence . . . . $x=15.02407+$

(5.) Given $x^3+x^2=500$, to find one value of x.

Here $A=1$, $B=0$, $r=7$.

B	0	
$r(r+A)$	56	
1st Divisor	56	500 (7.61
r^2	49	392
B'	161	108
$(3r+s+A)s$	1356	104736
2d Divisor	17456	3264
s^2	36	1887181
	18848	)1376819
	2381	Continue by common division.
3d Divisor	1887181	
	1	
	1889563	

(6.) Given $x^3+10x^2+5x=2600$, to find one value of x.

Here $A=10$, $B=5$, $r=11$.

B	5	2600 (11.006
$r(r+A)$	231	2596
1st Divisor . . .	236	4
r^2	121	3529188216
B'	588	470811784
$(3R+u)u$	198036	
4th Divisor . . .	588198036	Continue by common division.

The four miscellaneous examples on page 262, the last page of the school edition.

(1.) Let $x+y$ represent the greatest extreme, and $x-y$ the least extreme; then $\dfrac{x^2-y^2}{x}=$ the harmonical mean.

By the conditions of the problem we have

$$2x+\frac{x^2-y^2}{x}=26 \cdots\cdots (1) \quad \text{and} \quad \frac{(x^2-y^2)}{x}=576 \cdots (2)$$

By reducing (1), and taking the square root of equation (2), we have

$$2x^2+x^2-y^2=26x \cdots\cdots\cdots (3)$$

$$x^2-y^2=24\sqrt{x} \cdots\cdots\cdots (4)$$

By subtraction . . . $2x^2= \quad 26x-24\sqrt{x}$

If we put $\sqrt{x}=v$, and reduce the resulting equation, we shall have . . . . . . $v^3-13v=-12$

By (Art. 163) we find $v=3$. Hence $x=9$, &c.

(2.) Let $x+y=$ the greater number, and $x-y=$ the less. Then by the problem we have $x=5$

And . . . . $8x^3y+8xy^3=1040$

Or . . . . $y^3+25y=26$ Hence . . . . . . $y=1$

(3.) Let x, y, and z represent the numbers; then by the problem . . . . . $x+y+z=28 \cdots\cdots\cdots (1)$

$$x+yz=51 \cdots\cdots\cdots (2)$$

$$y+xz=87 \cdots\cdots\cdots (3)$$

By adding (2) and (3) we have $(x+y)+(x+y)z=138$

But by equation (1), $x+y=28-z$; therefore

$$28-z+28z-z^2=138 \quad \text{or} \quad \cdots\cdots\cdots z=5$$

(4.) Let x, y, and z represent the numbers; then

$$x^2+y^2+z^2=195 \cdots\cdots\cdots\cdots (1)$$

$$x^3+y^3+z^3=1799 \cdots\cdots\cdots\cdots (2)$$

$$xyz=385 \cdots\cdots\cdots\cdots (3)$$

The form of these equations will not be changed by taking x for z or x for y. A full and formal solution of these equations would become tedious, and to avoid this, we may venture a solution by inspection. As 195 and 385 both have 5 in the unit's place, it is more than probable that the value of one of the symbols is 5. Therefore assume $z=5$. This gives $x^2+y^2=170$, and $xy=77$, from which we find $x=7$ or 11, and $y=11$ or 7, and as 5, 7, and 11 will verify equation (2), this is a true solution.

The following examples are in the University Edition only.

(6.) Find one value of x from $5x^3-6x^2+3x=-85$.

As the result is negative, we will change the second and every alternate sign of the equation, (Art. 178), and find a value of x from the equation $5x^3+6x^2+3x=85$

Use the formula of (Art. 194.) $c=5$, $\mathcal{A}=6$, $B=3$, and by trial we find $r=2$.

		r s
$B \cdots\cdots\cdots$	3	85 (2.1
$(cr+\mathcal{A})r \cdots$	32	70
1st divisor ..	35	15
$cr^2 \cdots\cdots\cdots$	20	9.065
	87	5 935
$(3cr+cs+\mathcal{A})s$ ..	3.65	
2d Divisor ...	90.65	
$cs^2 \cdots\cdots\cdots$	5	
	94.35	

Continuing this we shall find the value of x to be 2.16399+, and its sign changed will be the value of x in the original equation.

(7.) Find x from the equation $12x^3+x^2-5x=330$

Here $c=12$, $A=1$, $B=-5$, $r=3$.

B	-5		
$(cr+A)r$	111		
1st Divisor	106		$330\ (\overset{r\ s\ t}{3.036}$
cr^2	108		318
B'	325		12
$(3cR+ct)t$		11208	9783624
3d Divisor		3261208	2216376
ct^2		108	
		3272524	Continue thus.

In the same manner perform (8) and (9.)

(Art. 195.) Page 323.

(3.) Extract the cube root of 1·352·605·460·594·688.

For the sake of brevity, take $r=11$, in place of 1.

1st Divisor . . .	121		$r\ s\ t$
$B'=3r^2$ . .	363	1·352·605·460·594·688	(110592
$(3R+t)t$ . .	16525	1 331	
2d Divisor	3646525	21 605 460	
	25	18 232 625	
	3663075	3 372 835 594	
$(3R+u)u$ . .	298431	3 299 453 379	
3d Divisor	366605931	73 382 215 688	
	81	73 382 215 688	
	366904443		
$(3R+v)v$ . .	663544		
	36691108844		

(4.) By a table of cubes which run to 8000, we perceive at once that r in this example is 17.

1st Divisor	289		*r s t*
$3r^2=B'$	867		5382674 (175.2
$(3R+s)s$	2575		4913
2d Divisor	89275		469674
	25		446375
	91875		23299000
$(3R+t)t$	10504		18396008
3d Divisor	9198004		4902992
	4		
	9208512		

Complete another divisor, then continue as in simple division.

It is not important to show a solution to the remaining examples under this article.

Page 324.

(**1.**) Change all the signs, then

$x^4+x^3+x^2-x=500$ $\quad r=4$

1	1	1	-1	500 (4
	4	20	84	332
	5	21	83	168
	4	36	228	
	9	57	311	
	4	52		
	13	109		
	4			
	17			
1	17	109	311	168 (.4 &c.

First Trans.

(**2.**) Resolved in the same manner as (**1.**)

(**3.**) The signs of all the terms must be changed, then

First Trans.

					r
1	—9	—11	—20	—4	(.1
	.1	—.89	—1.189	—2.1189	
	—8.9	—11.89	—21.189	—1.8811	
	1	—88	—1 277		
	—8.8	—12.77	—22.466		
	1	—87			
	—8.7	—13.64			
	1				
					s
1	—8.6	—13.64	—22.466	—1.8811	(.07 &c.

(4.) $x^5=5000$. Here all the coefficients are zero except the first, and $r=5$.

						r
1	0	0	0	0	=5000	(5
	5	25	125	625	3125	
	5	25	125	625	1875	
	5	50	375	2500		
	10	75	500	3125		
	5	75	750			
	15	150	1250			
	5	100				
	20	250				
	5					
	25					
						s
1	25	250	1250	3125	=1875	(.4 &c

(5.) $x^3=\dfrac{64x^2}{x^4+2x^2+1}$ or $x^5+2x^3+x=64$

						r
1	0	2	0	1	=64	(2
	2	4	12	24	50	
	2	6	12	25	14	
	2	8	28	80		
	4	14	40	105		

Continue as in last example.

In the editions of Algebra which were published prior to 1857, some few important problems were passed over in the Key without a solution.

They were purposely omitted, as an experiment, to discover whether there would be any special call upon the author for solutions; and the experiment is now over. The call for their solution has been very frequent and earnest, from all parts of the country: and this demonstrates the importance of a Key.

We now insert them, and some others which have been added to the enlarged edition, first published in 1858.

A more general Key to mathematical science is published in our Operations. In that work, the advanced student and the amateur mathematician will find many curious and useful problems, not merely in Algebra, but in Geometry and the higher sciences.

(Art. 107.) EXAMPLE 18.

Given . . . . $x-1=2+\frac{2}{\sqrt{x}}$

Place . $\sqrt{x}=y$. Then . $y^2-1=2+\frac{2}{y}$

$$y(y^2-1)=2(y+1)$$

Divide by $(y+1)$ then $y(y-1)=2$ or $y^2-y=2$.

Whence, ... $y=2$ or 1.

(Art. 114.) EXAMPLE 17.

Let x and y represent the number. Then, by the given conditions, we have:

$$x+y=xy \qquad (1)$$

$$\text{and} \quad xy=x^2-y^2 \qquad (2)$$

Assume $x=vy$. Then $vy+y=vy^2$, or $y=1+\frac{1}{v}$ (3)

and $vy^2=v^2y^2-y^2$

$v=v^2-1$ $\qquad v=\frac{1\pm\sqrt{5}}{2}$

Whence, . $y=1\pm\frac{2}{1\pm\sqrt{5}}=\frac{3\pm\sqrt{5}}{1\pm\sqrt{5}}$

But, . . . $x=vy=\frac{3\pm\sqrt{5}}{1\pm\sqrt{5}}\cdot\frac{1\pm\sqrt{5}}{2}=\frac{1}{2}(3\pm\sqrt{5})$

$$y=\frac{3+\sqrt{5}}{1+\sqrt{5}}\cdot\frac{1-\sqrt{5}}{1-\sqrt{5}}=\frac{-2-2\sqrt{5}}{-4}=\frac{1}{2}(1+\sqrt{5}).$$

Example 31. (Page 189.)

Let x and y represent the number. Then, by the given conditions:

$$xy=x^2-y^2 \qquad (1)$$

$$x^2+y^2=x^3-y^3 \qquad (2)$$

Assume $x=vy$. And this value of x, placed in (1) and (2), produce

$$vy^2=v^2y^2-y^2 \qquad (3)$$

and $$v^2y^2+y^2=v^3y^3-y^3 \qquad (4)$$

Dividing (3) and (4) each by y^2, and we obtain

$$v=v^2-1 \qquad (5)$$

and $$v^2+1=(v^3-1)y \qquad (6)$$

Observe that (5) is a quadratic, and its solution gives

$$2v=1\pm\sqrt{5} \qquad (7)$$

Again: Add 2 to each member of (5), and we have

$$v+2=v^2+1.$$

Also, multiply (5) by v, and

$$v^2=v^3-v, \quad \text{or} \quad v^2+v=v^3.$$

But . . . . $v^2=v+1$;

Whence, . $2v+1=v^3$, and $2v=v^3-1$.

Now, Equation (6) becomes

$$2vy=v+2,$$

or $$4vy=2v+4.$$

Substituting the value of $4v$ and $2v$, as found in (7), we have

$$2(1\pm\sqrt{5})y=5\pm\sqrt{5}$$

$$2y=\frac{5\pm\sqrt{5}}{\sqrt{5}\pm1}=\pm\sqrt{5} \qquad y=\pm\tfrac{1}{2}\sqrt{5} \quad \textit{Ans.}$$

Lastly, $x=vy=\frac{1}{2}(1\pm\sqrt{5})\frac{1}{2}\sqrt{5}=\frac{1}{4}(\sqrt{5}\pm5)$ *Ans.*

Another solution may be found in our Mathematical Operations, pages 115 and 116.

(Art. 204.) EXAMPLE 1.

$$x^5-5x^3+5x^2-1=0.$$

Here the sum of the coefficients is zero, therefore one root is $+1$, and the equation is divisible by $x-1$.

Dividing by $(x-1)$, the quotient is

$$x^4+x^3-4x^2+x+1=0.$$

Here, again, the sum of the coefficients is zero, showing that another root is $+1$; and this, again, is divisible by $(x-1)$. The next quotient is

$$x^3+2x^2-2x-1=0.$$

Here, again, the sum of the coefficients is zero, showing a third root equal to $+1$; and hence, we divide again by $x-1$. The quotient is

$$x^2+3x+1=0. \qquad \text{Whence, } x=-\tfrac{1}{2}(3\pm\sqrt{5}).$$

N. B.—We might have taken $x^4+x^3-4x^2+x+1=0$, and solved it by the rule (under Art. 203).

2. $$x^4+5x^3+2x^2+5x+1=0.$$

By the rule (under Art. 203) we have

$$x^4+5x^3+(2+\tfrac{25}{4})x^2+5x+1=\tfrac{25}{4}x^2;$$

Extracting square root, and

$$x^2+\tfrac{5}{2}x+1=\pm\tfrac{5}{2}x$$

$$x^2=-1, \qquad \text{or} \qquad x=\pm\sqrt{-1};$$

Or, ... $x^2+5x=-1 \qquad x=\tfrac{1}{2}(-5\pm\sqrt{21}).$

3. $$x^4-4x^3+4x-1=0.$$

$$(x^4-1)-4x(x^2-1)=0.$$

Dividing by (x^2-1), and $x^2+1-4x=0$, or $x^2-1=0$,
Or . . $x=\pm1$; or $x^2-4x+4=3$. $x=2\pm\sqrt{3}$.

4. $$x^4-\tfrac{5}{2}x^3+2x^2-\tfrac{5}{2}x+1=0.$$

By the rule (Art. 203), add $\tfrac{25}{16}x^2$ to each member—then $x^4-\tfrac{5}{2}x^3+\tfrac{57}{16}x^2-\tfrac{5}{2}x+1=\tfrac{25}{16}x^2$

Square root, $x^2-\tfrac{5}{4}x+1=\pm\tfrac{5}{4}x,$

Whence, . . $x^2+1=0$, or $x^2-\frac{5}{2}x+1=0$

$$x=\pm\sqrt{-1}, \quad\text{or}\quad x=2 \quad\text{or}\quad \tfrac{1}{2}.$$

5. $$5x^4+8x^3+9x^2+8x+5=0.$$

$$x^4+\tfrac{8}{5}x^3+\tfrac{8}{5}x+1=-\tfrac{9}{5}x^2$$

Add . . . . $$2x^2 \qquad =+\tfrac{10}{5}x^2$$

and $$x^4+\tfrac{8}{5}x^3+2x^2+\tfrac{8}{5}x+1=\frac{x^2}{5}$$

Add . . . . $$\frac{16x^2}{25} \qquad =\frac{16x^2}{25}$$

$$x^4+\tfrac{8}{5}x^3+\frac{66x^2}{25}+\tfrac{8}{5}x+1=\tfrac{21}{25}x^2$$

Square root, $$x^2+\tfrac{4}{5}x+1=\tfrac{1}{5}x\sqrt{21}$$ a quadratic.

$$x^2+\left(\frac{4-\sqrt{21}}{5}\right)x=-1$$

Place . . . . $$\frac{4-\sqrt{21}}{5}=2a \qquad 4-\sqrt{21}=10a,$$

Then . . . . $$x^2+2ax+a^2=a^2-1$$

$$x+a=\pm\sqrt{a^2-1}$$

$$x=-a\pm\sqrt{a^2-1}$$

Because $a^2=-\dfrac{8\sqrt{20-5}}{100}$ $\sqrt{a^2-1}$ is imaginary, and, therefore, the values of x are imaginary.

6. $$4x^6-24x^5+67x^4-73x^3+57x^2-24x+4=0.$$

Divide by x^3; then

$$4x^3-24x^2+57x-73+\frac{57}{x}-\frac{24}{x^2}+\frac{4}{x^3}=0 \qquad (1)$$

Assume $x+\dfrac{1}{x}=y$, Then $57x+\dfrac{57}{x}=57y$,

$$x^2+2+\frac{1}{x^2}=y^2 \qquad -24x^2-\frac{24}{x^2}=48-24y^2$$

Also, $x^3+\dfrac{1}{x^3}+3\left(x+\dfrac{1}{x}\right)=y^3$, or $x^3+\dfrac{1}{x^3}=y^3-3y$,

and $$4x^3+\frac{4}{x^3}=4y^3-12y.$$

These values placed in (1)

And $$4y^3-12y+48-24y^2+57y-73=0$$

Or $$4y^3-24y^2+45y-25=0, \qquad (2)$$

Here the sum of the coefficients is zero. Therefore, one value of y is 1.

Then $x+\frac{1}{x}=1$. Whence $x=\frac{1\pm\sqrt{-3}}{2}$

Dividing (2) by $y-1$, and we obtain

$$4y^2-20y+25=0$$

Square root $2y-5=0$, or $y=\frac{5}{2}$.

Whence $x+\frac{1}{x}=\frac{5}{2}$, and $x=2$ or $\frac{1}{2}$.

7. $$4x^4+3x^3-8x^2-3x+4=0. \qquad (1)$$

Here the sum of the coefficients is zero. Therefore, $x=1$ for one root. Again, if we change the second and every alternate sign, we shall have

$$4x^4-3x^3-8x^2+3x+4=0,$$

and the sum of the coefficients is still zero; therefore, $x=1$ in this equation, which corresponds to -1 in the given equation.

Therefore, (1) has two roots, $x=1$ and $x=-1$: hence, that equation is divisible x^2-1.

The quotient is, $4x^2+3x-4=0$,

Whence $$x=\frac{-3\pm\sqrt{73}}{8}$$

8. $$x^4+24x^3-2x^2-24x+1=0.$$

This, like the preceding, and for the same reason, is divisible by x^2-1.

The quotient is, $x^2+24x-1=0$,

Whence $x=-12\pm\sqrt{145}$.

9. $$x^4-2x^3-7x^2-8x+16=0. \qquad (1)$$

Here the sum of the coefficients is zero. Hence, $x=1$ for one root, and the equation can be depressed to

$$x^3-x^2-8x-16=0. \qquad (2)$$

Assume $x=nP$. Then

$$n^3P^3-n^2P^3-8nP-16=0$$

Let $n=4$; then $64P^3-16P^2-32P-16=0$

Divide by 16, $4P^3-P^2-2P-1=0$.

Here the sum of the coefficients is zero. Therefore, $P=1$. But $x=nP$, and $n=4$, $P=1$. Whence, $x=4$, for another root of the equation.

Now (2), divided by $x-4$, produces

$$x^2+3x+4=0,$$

Whence $$x=\frac{-3\pm\sqrt{-7}}{2}\quad\text{Imaginary.}$$

ANOTHER SOLUTION:

The attempt to extract square root results as follows:

$$\begin{array}{rl}
 & x^4-2x^3-7x^2-8x+16\ (x^2-x \\
 & \underline{x^4} \\
 & \quad -2x^3-7x^2 \\
2x^2-x & \quad \underline{-2x^3+\ x^2} \\
 & \qquad\quad -8x^2-8x
\end{array}$$

If the remainder were $+8x^2$ in place of $-8x^2$, this expression would be a square.

It will be $8x^2$, if we add $+16x^2$ to each member. Then we shall have

$$x^4-2x^3+x^2+8x^2-8x+16=16x^2,$$

Or . . . $$(x^2-x)^2+8\,(x^2-x)+16=16x^2,$$

Square root $$x^2-x+4=\pm4x.$$

10. $$x^4+2x^3-3x^2-4x+4=0. \qquad (1)$$

Here one value of x is 1. Dividing by $x-1$, we obtain

$$x^3+3x^2-4=0.$$

Here, again, $x=1$; and another division produces

$$x^2+4x+4=0,$$

Square root . . $(x+2)\ (x+2)=0$, $x=-2$ $x=-2$.

11. $$x^4-2x^3-25x^2+26x+120=0. \qquad (1)$$

Assume $x=nP$. Then

$$n^4P^4-2n^3P^3-25n^2P^2+26nP+120=0$$

$$P^4-\frac{2P^3}{n}-\frac{25}{n^2}P^2+\frac{26}{n^3}P+\frac{120}{n^4}=0.$$

Let $n=3$. Then $P^4-\frac{2}{3}P^3-\frac{25}{9}P^2+\frac{26}{27}P+\frac{40}{27}=0$,

Or $$27P^4-18P^3-75P^2+26P+40=0.$$

Here the sum of the coefficients is zero; therefore, $P=1$. But $n=3$, whence $x=3$.

If in (3) we assume $n=5$, the coefficients will again be zero, showing that another root is 5.

Now, equation (1) divided by $x-3$ or $x-5$, or their product, will produce a quadratic which will give two other roots, $x=-2$ and $x=-4$.

12. $$x^4-2x^3+2x^2-x=0. \quad (1)$$
$$(x^3-1)x=2x^2(x-1).$$

As x is a common factor, $x=0$; and $(x-1)$ is also a common factor, therefore $x=1$. Dividing by these factors, we obtain $x^2+x+1=2x$,

Or $$x^2-x=-1. \quad \text{Whence } x=\frac{1\pm\sqrt{-3}}{2}$$

13. $$x^4-4x^3+8x^2-32x=0. \quad (1)$$
Or . . . . . $(x-4)x^3+(x-4)8x=0$, Whence $x=0$,
Or . . $x=4$, or $x^2+x=0$, or $x=\pm\sqrt{-8}$.

14. $$x^3+5x^2+3x-9=0. \quad (1)$$
Here the sum of the coefficients is zero; therefore, $x=1$. And by division we obtain
$$x^2+6x+1=0, \quad \text{Whence } x=-3\pm\sqrt{8}.$$

15. $$x^3+6x^3-7x-60=0. \quad (1)$$
Assume $x=nP$. Then
$$n^3P^3+6n^2P^2-7nP-60=0.$$
Suppose $n=3$. Then
$$3)27P^3+54P^2-21P-60=0$$
$$9P^3+18P^2-\ 7P-20=0.$$

Here the sum of the coefficients is zero; therefore, $x=3$, for one of the roots of the equation. The other roots are -4 and -5.

16. $$x^3+8x^2+17x+10=0. \quad (1)$$
Change the 2d and each alternate sign, then
$$x^3-8x^2+17x-10=0.$$

Here the sum of the coefficients is zero. Hence, $x=1$ for the last equation; or, $x=-1$ for the given equation. Now divide the given equation by $x+1$, and the quotient will be $x^2+7x+10=0$, $x=-2$ or -5.

17. $$x^3-29x^2+198x-360=0. \quad (1)$$

Place $x=nP$. Then

$$P^3-\frac{29P^2}{n}+\frac{198P}{n^2}-\frac{360}{n^3}=0,$$

Assume $n=3$. Then

$$P^3-\frac{29}{3}P^2+22P-\frac{40}{3}=0$$

$$3P^3-29P^2+66P-40=0$$

Here the sum of the coefficients is 0. Therefore, $x=3$; and, dividing the equation by $(x-3)$, we obtain

$$x^2-26x+120=0,$$

$$x=6 \quad \text{or} \quad 20.$$

18. $$4x^3-112x^2+109x-27=0. \quad (1)$$

Here the absolute term -27 is not numerically large enough to make the sum of the coefficients zero; therefore, we take an operation which will increase it:

Place $x=\frac{P}{2}$ Then

$$\frac{4P^3}{8}-\frac{112P^2}{4}+\frac{109P}{2}-27=0$$

Multiply by 2, and $P^3-56P^2+109P-54=0. \quad (2)$

Here the sum of the coefficients is zero. Therefore, $P=1$. Whence $x=\frac{1}{2}$.

Dividing (2) by $P-1$, we obtain

$$P^2-55P+54=0.$$

Here, again, $P=1$; and, of course, another root of the original equation is $\frac{1}{2}$, and it has two equal roots.

Dividing (1) by $x-\frac{1}{2}$, and we obtain

$$4x^2-110x+54=0,$$

Whence . . . . $x=27$

ON INDETERMINATE EQUATIONS.

For the complete solution of a problem, we must have as many independent equations as unknown quantities to be determined.

When this is not the case, the problem is indeterminate. For example, $x+y=20$. x may be one or two or three, or $\frac{1}{2}$, or any other number, whole or fractional, under 20, and y will take the remaining part of 20, and the equation is indeterminate in the strictest sense of the term.

If, however, we restrict the values of x and y, to whole or *integer* numbers in the equation $x+y=20$, x cannot have more than 20 different values, when, without this restriction x might take an infinite number of values, and still preserve the equation $x+y=20$.

In some cases, the number of answers to an equation may be infinite, and the particular values restricted to integers. The following is a general case of the kind

$ax-by=c$. This gives $x=\frac{c+by}{a}$, where y may be any value whatever, that will give $\frac{c+by}{a}$, a whole number, but numberless such values of y can be found, as $c+by$ is a magnitude that can rise higher and higher, without limit, according to the assumed value of y.

But take y what we will, and the equation still exists, and therefore, the *number of answers for x is unlimited or infinite.* In such equations, *if the least values of x* and y are required, we have definite problems.

In equations like the following, $ax+by=c$, the number of answers in integer numbers, *may* be *very limited*, may be only one, or may be impossible. The equation gives

$$x=\frac{c-by}{a}.$$

Now, if c is very large, and b and a small, y *may take* many different values before $c-by$ is so small that we cannot divide it by a, and obtain an integer quotient.

When c is not large in reference to b and a, we may obtain only one value of y and x, and if by making $y=1$, we find $\frac{c-b}{a}$, a proper fraction, the problem is impossible.

EXAMPLE.

$3x+5y=13$. $x=\frac{13-5y}{3}$. If we take $y=1$. $x=\frac{8}{3}$, not an integer, therefore, y must not be taken equal to one. Take $y=2$, then $x=1$, both integer values, and the only integer values that will answer the conditions of the equation. $3x+5y=6$, is an equation in which it is impossible to give integer values to both x and y, because $3+5$ is more than 6, or $a+b$ greater than c or $\frac{c-b}{a}$ is a proper fraction.

The equation $ax+by=c$, is always possible in *integers*, when c is greater than $(ab-a-b)$, and a and b prime to each other. The equation *is sometimes* possible, when c is not greater than $(ab-a-b.)$

The equation $7x+13y=71$, is impossible in integers, for both x and y. But the equation $7x+13y=27$, *is possible*, $x=2$ and $y=1$. Here a and b are the same in both equations. In the first equation, $c=71$; c is not large enough to make the equation possible, for c, 71, is

not greater than $(7\cdot 13-20)$, but if c was *any number* greater than 71, the equation would be possible in integers, and it is possible with *some numbers* less than 71.

If a and b are not prime to each other in the equation $ax+by=c$, c must be divisible by the same number that divides a and b, or the equation is *impossible* in integers for x and y.

EXAMPLES.

1. Given $6x+9y=32$. Or, $2x+3y=\frac{32}{3}$.

But if x is a whole number, $2x$ will be a whole number, and by the same considerations, $3y$ must be a whole number, and two or more *whole numbers added together can never make a fraction*, therefore the equation $2x+3y=\frac{32}{3}$, or $6x+9y=32$, is impossible in integers.

In cases where solutions are possible, our rules of operation rest entirely on these considerations:

1st. *A whole number added to a whole number, the sum is a whole number.*

2d. *A whole number taken from a whole number, the remainder is a whole number.*

3d. *Multiply a whole number by a whole number, and the product is a whole number.*

For instance, if x is a whole number, $2x$, $3x$, $4x$, or any integral number of x is a whole number.

2. Given $3x+5y=35$, to find x and y in whole numbers. $x=\frac{35-5y}{3}$. But as x is a whole number, its equal, or $\frac{35-5y}{3}$, must equal *some whole number.*

But $\frac{35-5y}{3}=11-y+\frac{2-2y}{3}$.

Now $11-y$ being a whole number take it away, and the remainder $\frac{2-2y}{3}$, must also be a whole number.

But y being a whole number, $\frac{3y}{3}$ is a whole number,

Therefore, $\frac{3y}{3}+\frac{2-2y}{3}=\frac{y+2}{3}$ is a whole number, which number call p. And $\frac{y+2}{3}=p$. Or, $y=3p-2$.

For the least value of y make $p=1$, and y will equal 1, and $x=\frac{35-5y}{3}=10$. Make $p=2$, then $y=4$, and $x=5$. Make $p=3$, then $y=7$, and $x=0$.

Hence, $y=1$ or 4, and $x=10$ or 5, are the only results this equation admits of.

In operating on the fractional expression $\frac{35-5y}{3}$, it was *our object* to *work down* the coefficient of y to 1. To accomplish this object, we cast out whole numbers, add and subtract whole numbers *in the shape of fractions*, &c., only taking care to keep expressions that are equal to integers, until the coefficient of y becomes one.

3. Given $35x-24y=68$, to find the *least values* of x and y in integers.

The number of values in this case is unlimited.

$$x=\frac{68+24y}{35}=1+\frac{33+24y}{35}.$$

Hence, $\frac{33+24y}{35}=$ some whole number; but $\frac{35y}{35}$ is also a whole number. $\frac{35y}{35}-\frac{33+24y}{35}=\frac{11y-33}{35}=$ an integer.

$$\text{Or, } \frac{33y-99}{35}=\frac{33y-29}{35}-2.$$

$\frac{35y}{35}-\frac{33y-29}{35}=\frac{2y+29}{35}=$ a whole number, multiply by 18, and $\frac{36y+18\cdot 29}{35}=y+14+\frac{y+32}{35}=$ a whole number.

Therefore $\frac{y+32}{35}=p$. Or, $y=35p-32$.

Take $p=1$ for the least value of y, and $y=3$. Therefore $x=4$.

4. A man wishes to lay out $500 for cows and sheep: the cows at 17 dollars per head, and the sheep at 2 dollars. How many of each did he purchase?

Let $x=$ the number of cows, and y the number of sheep. Then $17x+2y=500$. We know this equation is restricted to whole numbers; because the man could not have part of a cow, or part of a sheep.

To find the least number of cows, transpose $17x$.

Then $$y=250-8x-\frac{x}{2}.$$

Now as y must be a whole number, and $250-8x$ must also equal some whole number, $\frac{x}{2}$ must be a whole number. That is, the number of cows must be *an even number*; because the number must be divisible by 2.

Hence, $\frac{x}{2}=p$. Or, $x=2p$. Make $p=1$, and $x=2$, the least number of cows. Then $y=233$, the corresponding number of sheep.

Now if the man wished to purchase as few sheep, and as many cows as possible, we should transpose the other term, thus: - - - - - $x=\frac{500-2y}{17}=29+\frac{7-2y}{17}$.

Therefore, $\frac{7-2y}{17}=$ a whole number. Multiply by 8, and $\frac{56-16y}{17}=$ a whole number, to which add $\frac{17y}{17}$ and we have $\frac{56+y}{17}=3+\frac{5+y}{17}$. Drop off the whole number 3, then $\frac{5+y}{17}=p$. Or, $y=17p-5$. Making $p=1$, gives $y=12$, the smallest number of sheep. This gives $x=28$, the corresponding number of cows.

The number of cows or x, may be any one of the even numbers from 2 to 28.

5. A man wished to spend 100 dollars in cows, sheep, and geese; cows at 10 dollars a piece, sheep at 2 dollars, and geese at 25cts., and the aggregate number of animals to be 100. How many must he purchase of each?

Let $x=$ the number of cows, y the sheep, and z the geese.

Then - - - $10x+2y+\frac{z}{4}=100.$ (1)

And - - - - $x+y+z=100.$ (2)

Clear equation (1) of fractions, and

$$\begin{array}{r} 40x+8y+z=400. \\ x+\ y+z=100. \\ \hline 39x+7y\quad\ =300. \end{array}$$

$$x=\frac{300-7y}{39}=7+\frac{27-7y}{39}=\text{a whole number.}$$

Or, $5\left(\frac{27-7y}{39}\right)=\frac{135-35y}{39}=$ a whole number, add $\frac{39y}{39}$ and $\frac{4y+135}{39}$, or $\frac{40y+1350}{39}=y+34+\frac{y+24}{39}=$ a whole number.

Therefore, $\frac{y+24}{39}=p.$ Or, $y=39p-24=15.$

This value of y, gives $x=5.$ Hence, $z=80.$

If we take $p=2$, we shall have $y=54$; then x will come a minus quantity an inadmissible circumstance in any problem like this. Therefore, 5 cows, 15 sheep, and 80 geese, is the only solution.

6. A person spent 28 shillings in ducks and geese; for the geese he paid 4s. 4d. a piece, and for the ducks, 2s. 6d. a piece. What number had he of each?

Let $x=$ the number of geese, and y the number of ducks.

Then. $52x+30y=28\cdot12.$ Or, $26x+15y=168.$

We will now show another operation to reduce this kind of equations.

Take the lowest coefficient, (in this example it is 15,) and observe whether it will divide the other numbers or not. If

it will divide, *reserve* the whole numbers, and omit the fractions. In this case, 15 *will not* divide 26, and will divide 168. The quotient will be 11, *disregard* the remainder.

Now assume $p=x+y-11$. Then, since x and y are whole numbers, p must be a whole number. Multiply by 15, and transpose, and we have

$$15p-15x-15y=-165.$$
$$26x+15y=\ \ 168.$$

By addition, $15p+11x\quad =3.$ Assume $q=x+p$.
Then - - - - $11q-11p-11x=0.$
Sum - - - - $11q+\ 4p\quad =3.$ Assume $r=p+2q$
Or, - - - - - - $4r-\ 4p-\ 8q=0.$
Sum - - - - - $4r\quad +\ 3q=3.$ Assume $s=r+q-1$
Or, - - - - - - $3s-\ 3r-\ 3q=-3.$
Sum, - - - - - $3s+\ r\quad =0.$ Or, $r=-3s.$

Now having worked down to unity for a coefficient, the problem is essentially reduced. We can make $s=0$, hence $r=0$. Then in the last assumed equation $q=1$, and in the equation, $r=p+2q$, gives $p=-2$; and in the preceding assumed equation, $q=x+p$, that is $x=3$, and the first assumed equation gives $y=6$.

Hence 3 geese and 6 ducks is the answer, and no other numbers will do.

7. Divide the number 100 into two such parts, that one of them may be divisible by 7, the other by 11.

Let $7x=$ one part, and $11y=$ the other.

Then $7x+11y=100$, and x and y must be whole numbers. Assume $p=x+y-14$. (1)

Then - - - - - - $7p-7x-\ 7y=-98.$
But - - - - - - - $7x+11y=\ 100.$
By addition, - - $7p\quad +\ 4y=2.$

Assume $q=p+y$. (2)

Then - - - - - - $4q-4p-\ 4y=0.$
Add, and - - - - $4q\quad +\ 3p=2.$

$$4q+3p=2.$$

Assume $r=q+p$. (3)

Then - - - - - - $3r-3q-3p=0.$

By addition - - - $3r+q \qquad =2.$ Or, $q=2-3r.$

Take $r=0$, then $q=2$, and $p=-2$. And from equation (2), $2=-2+y$, or $y=4$, and $11y=44$, one of the numbers, and of course 56 is the other,

8. Find a number which being divided by 6, shall leave the remainder 2, and the same number divided by 13 shall leave the remainder 3.

Consider that in division, the divisor and quotient multiplied together, and the remainder added, gives the number divided.

Let N represent the number divided, x and y the quotients.

Then $6x+2=N$, and $13y+3=N$.

Consequently, $6x-13y=1$, an equation in which x and y must be whole numbers, because they represent the whole numbers of the division.

Assume $p=x-2y$. We take $2y$ because 6 is contained in 13, twice.

Then - - - - - - $6p-6x+12y=0.$

And - - - - - - $6x-13y=1.$

Add, and - - - - $6p \qquad -y=1.$ Or, $y=6p-1.$

For the smallest value of y we must take $p=1$. Then $y=5$, and $13y+3=68$, the answer.

9. What number is that which being divided by 11, leaves a remainder of 3, divided by 19, leaves a remainder of 5, and divided by 29, shall leave a remainder of 10.

Let N be the required number, and x, y, and z the several quotients, and of course they must be whole numbers.

Then $11x+3=N$, and $19y+5=N$, and $29z+10=N$.

Hence, $x=\frac{29z+7}{11}$, and $x=\frac{19y+2}{11}$. $19y=29z+5$.

Or, - - - - - - - - - $19y-29z=5$.

Assume $p=y-z$. (1)

Then - - - - $19p-19y+19z=0$.

Add, and - - $19p \quad -10z=5$.

Assume $q=p-z$. (2)

Then - - - - $10q-10p+10z=0$,

By addition - $10q+9p \quad =5$.

Assume $r=q+p$. (3)

Then - - - - $9r-9q-9p=0$.

Add, and - - - $9r+q \quad =5$. Or, $q=5-9r$.

By returning to equations (3) and (2), we find $p=10r-5$, and $z=19r-10$. Not only must z be a whole number, but to make x a whole number, $\frac{29z+7}{11}$ must be a whole number. Substituting the value of z in this last expression, and we have $\frac{551r-283}{11}$ a whole number, or $50r-25+\frac{r-8}{11}$ a whole number. Therefore, $\frac{r-8}{11}$ must be a whole number, which call t; then $r=11t+8$. Let $t=0$, then $r=8$, $z=142$, and $29z+10=N=4148$, the number.

10. Required the least number that can be divided by each of the nine digits without remainders.

Let $x=$ the number.

Then $\frac{x}{2}$, $\frac{x}{3}$, $\frac{x}{4}$, $\frac{x}{5}$, $\frac{x}{6}$, $\frac{x}{7}$, $\frac{x}{8}$, $\frac{x}{9}$, must all be whole numbers.

Now if we make $\frac{x}{8}$ a whole number $\frac{x}{4}$ and $\frac{x}{2}$, the double, and quadruple will be whole numbers of course. Also if $\frac{x}{9}$ is a whole number, $\frac{x}{3}$ *will be* a whole number.

Therefore we have only to find such a value of x as will make $\frac{x}{9}$, $\frac{x}{8}$, $\frac{x}{7}$, $\frac{x}{6}$, $\frac{x}{5}$ whole numbers. $\frac{x}{6}$, may also be cast out, on consideration that $6=2 \cdot 3$, and 2 3 are factors, one of 9, the other of 8, in preceding expressions.

Hence we have only to make $\frac{x}{9}$, $\frac{x}{8}$, $\frac{x}{7}$, $\frac{x}{5}$, whole numbers. Put $x=9p$.

Then $\frac{9p}{8}=p+\frac{p}{8}$. Hence, $p=8q$. Then $x=9p=72q$. $\frac{72q}{7}=$ a whole number. $\frac{2q}{7}=$ a whole number, or $\frac{8q}{7}=q+\frac{q}{7}$. Make $\frac{q}{7}=r$, or $q=7r$. In the same way we find $r=5s$. Take $s=1$, then $r=5$. $q=35$. $x=72\cdot35=2520$.

N. B. By the aid of the Indeterminate analysis, combined with some well known properties of numbers, we may sometimes solve miscellaneous problems in a very summary manner. For example, we refer to the following.

11. The product of five numbers in arithmetical progression is 10395, and their sum is 35. What are the numbers?

As the number of terms is odd, let x represent the middle term, and y the common difference. Then $(x-2y)$, $(x-y)$, x, $(x+y)$, $(x+2y)$, will be the numbers, and their sum $5x=35$, or $x=7$.

Now as 10395 is the product of all the numbers, 7 must be one of its factors. Therefore divide by 7, and we have 1485 for the product of the four remaining terms, but as this number ends in 5, it can be divided by 5, and it is more than probable that 5 is one of the numbers, and the one preceding 7, therefore, 2 is the common difference, and 3, 5, 7, 9, and 11, the numbers.

12. Given the sum of the squares of three numbers $=195$, the sum of their cubes $=1799$, and their continued product $=385$, to find the numbers.

By the common rules, without considering any circumstances, this problem would produce equations of high order, and difficult of solution. But let us call to mind the fact, that *the numbers cannot be fractional*, if they were, their squares, cubes, and product *would not probably come whole numbers.* Also, some of the numbers must be under 10;

if even two of them were over 10, the cubes and product must be larger than the numbers mentioned. From considerations like these, we decide that the answer must be whole numbers, and some of them under 10. Now as the product of the three is 385, and the sum of their squares is 195, *both numbers* ending in 5, *it is so probable* that one of the numbers is 5, that we shall so consider it, and let x and y be the other two numbers, then $x^2+y^2+25=195$, and $5xy=385$. Equations from which we readily find one number to be 7, the other 11.

Now by trial we find $5^3+7^3+11^3=1799$, and therefore these *are in fact* the numbers.

SECTION XX.

To determine the number of solutions an equation may admit.

It has already been observed that an equation in the form of $ax-by=c$ admits of an infinite number of solutions, as $x=\frac{c+by}{a}$ a quantity all positive, and the only restriction is to assume y of such a value that the numerator may be divided by a.

But equations in the form of $ax+by=c$. Then $x=\frac{c-by}{a}$ and the numbers of solutions depends on the relative values of c, b, and a.

If c be very large in relation to b and a, as we have before observed, the equation may have many solutions, otherwise not.

We come now in a general manner to determine the number of solutions an equation of this form may have.

Let $ax+by=c$. Assume $ax'-by'=1$. Which equation

is always possible, and from which x' and y' can be known in integers.

Multiply the assumed equation by c, and $acx'-bcy'=c$. Put the two values of c equal to each other, and

$$ax+by=acx'-bcy'$$

$$x=cx'-b(y+cy'). \quad \text{Or, } x=cx'-bm. \qquad (1)$$

$$y=a\left(\frac{cx'-x}{b}\right)-cy'. \quad \text{Or, } y=am-cy'. \qquad (2)$$

In these theoretical equations, (1) and (2), m has different values, it being an arbitrary number taken at pleasure, so that cx' may be greater than bm, and am greater than cy' to render x and y positive.

But if no such value of m can be found, it is proof that values of x and y do not exist in positive integers, and on the contrary as many suitable values of m as can be found, so many solutions will the equation admit of, and no more.

Now as - - - $mb<cx'$, and $am>cy'$

Or, - - - - - $m<\frac{cx'}{b}$, and $m>\frac{cy'}{a}$

That is m at the same time is found to be greater and less than known quantities, therefore its *limit* or *range* is found.

For instance, if m must be greater than 30, and less than 40, we conclude that it may be any number between 30 and 40, and the *number of different values it can take* is 9.

We perceive that the difference between the integral parts of $\frac{cx'}{b}$ and $\frac{cy'}{a}$ will express the range of m, and the number of different solutions which the equation admits of, (except in certain cases;) as m is more than one of these fractions and less than the other, the difference between the expressions $\frac{cx'}{b}$ and $\frac{cy'}{a}$ is sometimes *one more* than the number of different values of m, such is the case when $\frac{cx'}{b}$ is an integer, in such cases, subtract one from the difference of these quantities for the range of m, but this case very seldom occurs.

N. B. In making use of the expression $\frac{cx'}{b}$ and $\frac{cy'}{a}$ care must be taken, not to take their difference as *factional expressions*, their *absolute difference is not wanted*, it is the difference between the *integral parts of* $\frac{cx'}{b}$ and $\frac{cy'}{a}$.

EXAMPLE.

Required the number of integral solutions to the equation $7x+9y=100$.

Find the least value to x', y' in the equation $7x'-9y'=1$.

$$7x'-9y'=1. \quad \text{Assume } p=x'-y'$$

Then - - $7p-7x'+7y'=0.$

Add, and $7p \qquad -2y'=1.$ Assume $q=3p-y'$

Then - - $2q-6p+2y'=0.$

Add, and $\qquad 2q+p=1.$ Or, $p=1-2q.$

Take $q=0$, then $p=1$. $y'=3$. $x'=4$.

Then $\frac{cx'}{b}=\frac{100\cdot4}{9}$ and $\frac{cy'}{a}=\frac{100\cdot3}{7}$.

That is $\frac{cx'}{b}=44\frac{4}{9}$ and $\frac{cy'}{a}=42\frac{6}{7}$.

Disregarding the fractions, the difference of the *integral parts* is 2, that is *there are two integral solutions to the equation.*

If we had taken the difference between $\frac{400}{9}-\frac{300}{7}$, *in a fractional form*, thus: $\frac{2800}{63}-\frac{2700}{63}=\frac{100}{63}=1\frac{37}{63}$.

Here the *integral difference* is one, which without this caution might be taken for the number of solutions. The *integral difference* in this case is not the *difference of the integrals.*

Observe in this example $44\frac{4}{9}$ and $42\frac{6}{7}$, the *fractional part* of $\frac{cx'}{b}$ is $\frac{4}{9}$, and the fractional part of $\frac{cy'}{a}$ is $\frac{6}{7}$, the former is less than the latter, in such cases the integral parts must be taken separately.

But when the fractional part of $\frac{cx'}{b}$ is not less than the fractional part of $\frac{cy'}{a}$, but equal or greater than it, we may find the number of solutions by taking the difference of the expressions $\frac{cx'}{b}-\frac{cy'}{a}$. Reduce to a common denominator, and take the difference of the numerators, and we will have $\frac{c\,(ax'-by')}{ab}$; but $ax'-by'=1$. Therefore, we have $\frac{c}{ab}$ for the number of solutions, at once.

Example.

What number of integral solutions will the equation $9x+13y=2000$ admit of? Ans. 17.

$$9\times 13=117)\,2000\,(\,17.$$

But as we cannot know whether the fractional part of $\frac{cy'}{b}$ is not less than the fractional part of $\frac{cy'}{a}$ we cannot be sure that dividing c by ab will give the true number of solutions. It either will be the true number *or one less.*

The equation $5x+9y=40$ admits of no solution in whole numbers, $c=40$ will not be divided by $ab=45$.

Now take the equation $5x'-9y'=1$. And we find $x'=2$, and $y'=1$.

Therefore, $\frac{cx'}{b}=\frac{80}{9}=8\frac{3}{9}$, and $\frac{cy'}{a}=\frac{40}{5}=8$. Now as there is no difference between these integral parts, it indicates as it should, that there is no solution.

But let us take $5x+9y=37$, the same equation, except a smaller value of c. If c would not divide by ab before, much less will it now. Yet in this last equation we have a solution. $x'=2$, $y'=1$, as before, and $c=37$, $\frac{cx'}{b}=\frac{74}{9}=8\frac{2}{9}$ and $\frac{cy'}{a}=\frac{37}{5}=7\frac{2}{5}$. Here the difference of the integrals is 1, and indicates one solution. $x=2$, $y=3$.

How many solutions will the equation $2x+5y=40$ admit of?

The auxiliary equation $2x'-5y'=1$, gives $x'=3$, $y'=1$. $\frac{cx'}{b}=24$. $\frac{cy'}{a}=20$. Or, 4 solutions.

But observe that $\frac{cx'}{b}$ in this case, is a complete integral, 24; agreeable then, to previous considerations we must deduct one, and the number of solutions are but 3, as follows: $x=5.\ 10.\ 15.$ $y=6.\ 4.\ 2$, and no other solution can be found.

What number of solutions in whole numbers can be found for the equation $3x+5y+7z=100$.

As x and y each cannot be less than one, z cannot be greater than $\frac{100-3-5}{7}=13\frac{1}{7}$. That is, z cannot be greater than 13, in whole numbers. Now suppose $z=1$, and the equation becomes $3x+5y=93$.

The number of solutions for this equation, found as previously directed is 6. That is $\left\{\begin{array}{l} x=26.\ 21.\ 16.\ 11.\ \ 6.\ \ 1. \\ y=\ \ 3.\ \ 6.\ \ 9.\ 12.\ 15.\ 18 \end{array}\right.$

Now x and y can make these 6 changes, and z be constantly equal to 1, *and satisfy the primitive equation.*

Take $z=2$, and the equation becomes $3x+5y=86$.

This equation has also 6 solutions, z being through all the changes of x and y equal to 2.

Now take $z=3$, then the original equation is $3x+5y=79$. This equation has five solutions.

Now take $z=4$, then $3x+5y=72$. This equation has four solutions.

Take $z=5$, then $3x+5y=65$. This equation has four solutions.

Take $z=6$, then $3x+5y=58$. This equation has four solutions.

In this manner, by taking z equal to all the integers up to 12 in succession, we find 41 solutions.

THE DIOPHANTINE ANALYSIS.

The Diophantine Analysis teaches how to find square and cube numbers under given conditions, or having given relations to each other.

EXAMPLES.

CASE 1ST. Find such a value of x as will make the expression $ax+b$ a square.

Put $ax+b=n^2$, n^2 being *any square*, it is, therefore, *an indefinite problem.*

From the equation $x=\frac{n^2-b}{a}$; take n equal to any number whatever, and a and b being known, x becomes known.

Eight times a certain number added to 9, makes a square. What is the number?

Let $x=$ the number. Then $8x+9=n^2$, *that is any square.* $a=8$, $b=9$, and $x=\frac{n^2-b}{a}=\frac{n^2-9}{8}$. Assume $n=7$, then $x=5$, the required number. But there are many other numbers that will answer the condition according as we assume n more or less.

Find x, such that the following expressions shall be square numbers.

$$9x+9. \quad 7x+2. \quad 3x-5. \quad x+\tfrac{1}{2}.$$

All these correspond to the general expression $ax+b$.

CASE 2D. Any algebraic expression in the general form of ax^2+bx, may be made a square by supposing its square root equal px; x must be in some part of the root, because the expression contains x^2, that is some function of x. Now if px is the root, $ax^2+bx=p^2x^2$. Divide by x, &c and we have $x=\frac{b}{p^2-a}$. p may be any assumed value whose square is greater than a.

EXAMPLES.

Six times the square of a certain number, added to five times the number is a square. What is the number?

$$6x^2+5x=p^2x^2. \quad \text{Or, } x=\frac{5}{p^2-6}.$$

Here it is obvious that p^2 must be greater than 6, otherwise it is unlimited. Take $p=3$, then $x=\frac{5}{3}=1\frac{2}{3}$.

Find the value of x to make $\frac{x^2}{2}+3x$ a square.

Ans. $x=6$.

CASE 3D. Any algebraic expression in the general form of $x^2\pm bx+c$, can be made a square, by putting $x\pm p$ equal its square root.

We can if we please take $x-p$ for the root in all such cases. Then if p is less than x, the square is diminished, if greater, the whole root will be essentially minus, but the square will be plus, and may rise to any amount. Therefore $x-p$ is far more general than $x+p$.

CASE 4TH. Any expression in the form of $ax^2\pm bx+c^2$, can be made a square, by taking its root equal to $c\pm px$.

It will be observed that x must be in the root of the previous expression, because it has x^2, and c must be in the root of this last expression, because it contains c^2.

In the first we have $x^2\pm bx+c=x^2\pm 2px+p^2$. Or, $x=\frac{p^2-c}{2p\pm b}$.

In the second, $x^2\pm bx+c^2=c^2\pm 2cpx+p^2x^2$. Or, $x\pm b=\pm 3cp+p^2x$. Or, $x=\frac{\pm 2cp\mp b}{1\pm p^2}$ In both cases assume p of any convenient value to render x positive, and as small as possible.

Find a number such, that if it be increased by 2 and 5 separately, the product of the sums shall be a square.

Let $x=$ the number, then $(x+2)(x+5)=x^2+7x+10$,

must be a square. $b=7$, $c=10$. General solution, $x=\frac{p^2-10}{2p\pm7}$

Now p^2 must be more than 10; hence, take $p=4$, and $x=\frac{6}{15}=\frac{2}{5}$, the *least* number that will answer the conditions.

CASE 5TH. An expression in the form of ax^2+bx+c, where *neither the first nor the last* terms of the expression are squares, neither branch of the root can be directly found, and the expression cannot be made a square, unless we can separate it into two *rational factors*, or unless we can first subtract from it some simple binomial square, and can then divide the remainder into two rational factors.

By reminding one of the nature of quadratic equations, all may perceive that the expression ax^2+bx+c *must be* the product of two factors, but whether *rational factors* or *not* is the subject of inquiry.

To find the factors which make the product ax^2+bx+c, put this expression equal to 0, and work out the values of x thus, $ax^2+bx+c=0$. Or, $ax^2+bx=-c$. Complete the square, and $4a^2x^2+4abx+b^2=b^2-4ac$.

$$\text{Or, } 2ax+b=\pm\sqrt{\{b^2-4ac.\}}$$

$$\text{Or, } x=\pm\frac{1}{2a}\sqrt{\{b^2-4ac\}}-\frac{b}{2a}.$$

We now perceive that the values of x must be rational, provided $\sqrt{\{b^2-4ac\}}$ is a complete square. If it be so, let $\frac{1}{2a}\sqrt{\{b^2-4ac\}}-\frac{b}{2a}=m$, and $-\frac{1}{2a}\sqrt{\{b^2-4ac\}}-\frac{b}{2a}=n$.

Then the two values of x are $x=m$ and $x=n$, and $(x-m)(x-n)$, are the factors which will give the expression ax^2+bx+c.

EXAMPLES.

1. Find such a value of x as will make $6x^2+13x+6$ a square.

Here $a=6$, $b=13$, $c=6$. $b^2=169$, $4ac=144$, $b^2-4ac=25$ and $\sqrt{\{b^2-4ac\}}=5$. Now $12x+13=\pm5$. $x=-\frac{2}{3}$, Or, $x=-\frac{3}{2}$. Or, $3x+2=0$, and $2x+3=0$.

That is, $(3x+2)(2x+3)$ will produce the expression $6x^2+13x+6$.

Now to make the expression a square, put

$$(3x+2)(2x+3)=p^2(3x+2)^2$$

Then $2x+3=p^2(3x+2.)$ And $x=\frac{2p^2-3}{2-3p^2}$.

Take $p=1$, and $x=1$. We might have seen at first, that in this *particular expression*, the value of x being 1, would make it a square, as $6+13+6=25$, a square.

That is in all cases when the sum of the coefficients make a square number, the value of x may be one.

2. Find such a value of x as shall render the expression $13x^2+15x+7$ a square.

Here as neither the first nor last terms are squares, nor b^2-4ac a square, the expression cannot be made a square, unless we can separate the remainder into factors after taking away some simple square. But in this case, $4ac$ is greater than b^2, we must then, in subtracting our square, diminish a and c *and increase* b.

To accomplish this end we will subtract the square of $x-1$, *not* $x+1$.

That is - - - - - -	$13x^2+15x+7.$
Subtract - - - - - -	$x^2-\ 2x+1.$
Remainder, - - - -	$12x^2+17x+6.$

In this last expression, $a=12$, $b=17$, $c=6$. Hence, $b^2-4ac=289-288=1$, a square; we are now sure rational factors can be found to produce the expression

$$12x^2+17x+6.$$

By assuming $12^2+17x+6=0$, and finding the values of x by the quadratic, (*merely to get the factors*,) we find $x=-\frac{2}{3}$, and $x=-\frac{3}{4}$. Or, $3x+2=0$, and $4x+3=0$.

And $(3x+2)(4x+3)=12x^2+17x+6$.*

* The values of x used to obtain these factors have no connection with the values of x to *render* the original expression a square.

Now the original expression is the same as $(x-1)^2+(3x+2)(4x+3.)$ Put its root equal to $(x-1)+p(3x+2,)$ and square this root, and

$(x-1)^2+(3x+2)(4x+3)=(x-1)^2+2p(x-1)(3x+2)+p^2(3x+2)^2$. By reduction, $4x+3=2p(x-1)+p^2(3x+2$. Or, $(-4+2p+3p^2)x=2p+3-2p^2$.

$$\text{Therefore, } x=\frac{2p+3-2p^2}{3p^2+2p-4}.$$

Take $p=1$, then $x=3$, and $13x^2+15x+7=169$ a square.

Ryan makes x in this example equal $\frac{1}{3}$ and the expression equal $\frac{121}{9}$ a square, but an integer number is always more satisfactory.

3. Find such a value of x as will render $14x^2+5x-39$, a square.

After a few trials this expression is found to be the same as $(2x-1)^2+(5x-8)(2x+5.)$ Assuming its root to be $2x-1+p(5x-8)$ Then by squaring the root, making it equal to its power, and reducing, we find $x=\dfrac{8p^2+2p+5}{5p^2+4p-2}$.

Assuming $p=1$, $x=\frac{15}{7}$, and the expression equals 36, a square. Other values can be found, by assuming different values to p.

4. Find such a value of x as shall make $2x^2+21x+28$ a square.

After a little inspection we find this expression equal to $(x+4)^2+(x+1)(x+12.)$ Now if we make

$$(x+4)^2+(x+1)(x+12)=\{(x+4)-p(x+1.)\}^2$$

After reduction, we shall find $x=\dfrac{12+8p-p^2}{p^2-2p-1}$.

Assume $p=4$, then $x=4$, and the original expression is 144 a square.

If $(x+4)^2+(x+1)(x+12)=\{(x+4)-p(x+12)\}^2$, we shall find $x=\dfrac{12p^2-8p-1}{1+2p-p^2}$. If we take $p=1$, $x=\frac{3}{2}$. If we

take $p=\frac{3}{2}$, $x=8$, and we might find many other numbers that would answer the conditions of the expression.

CASE 6TH. When we have an expression in the form of $a^2x^4+bx^3+cx^2+dx+e$, we can assign a value to x that will make the whole expression a square, *if we can extract three terms of its root.*

Assume such terms as the whole root, square the root so assumed, making it equal to the given expression, and by reducing, we shall have *a value of x* which will make the original expression a square.

EXAMPLE.

Find such a value of x as shall make $4x^4+4x^3+4x^2+2x-6$ a square.

We commence by extracting the square root as far as *three terms*, and find them to be $2x^2+x+\frac{3}{4}$. Now assume

$$4x\ +4x^3+4x^2+2x-6=(2x^2+x+\tfrac{3}{4}.)^2$$

Expanding and reducing, we have $2x-6=\frac{3x}{2}+\frac{9}{16}$.

And $x=13\frac{1}{8}$.

Essentially the same method must be pursued in other problems of the like kind.

CASE 7TH. Find such a value of x as shall render $2x^2+2$ a square.

Expressions of this kind, when neither a nor c are squares, nor b^2-4ac a square, and which cannot be resolved into factors, presents an impossible case, unless we can first find by inspection, some simple value of x that will answer the condition.

In the present example, it is obvious that if $x=1$, the expression is a square. But we wish to find other values than 1 that will render this expression a square, and having found that one will answer, we are now enabled to find other values, thus:

Let $x=1\pm y$. Then $x^2=1\pm2y+y^2$.

And $2x^2+2=4\pm4y+2y^2$. Here the original expression is transformed into another expression, *having a square* for its first term.

Now we must find such a value of y as shall make $4+4y+2y^2$ a square.

Assume $4+4y+2y^2=(2-my)^2=4-4my+m^2y^2$. Or, $4+2y=-4m+m^2y$. Hence $y=\frac{4(m+1)}{m^2-2}$, m may be any number greater than one. Put $m=2$. Then $y=6$, and $x=1+y=7$, and the original expression $2x^2+2=98+2=100$, a square.

N. B. It often occurs incidentally in the solution of problems, that we must make a square of two other squares. This can be done thus: Let it be required to make x^2+y^2 a square. Assume $x=p^2-q^2$, and $y=2pq$.

Then, - - - - - - $x^2=p^4-2p^2q^2+q^4$.

And, - - - - - - $y^2=\quad 4p^2q^2$.

Add, and - - $x^2+y^2=p^4+2p^2q^2+q^4$, which is evidently a square, whatever be the values of p and q. We can, therefore, assume p and q at pleasure, provided p be greater than q.

Double and Triple Equalities.

In the preceding section it was only necessary to find such a value of the unknown term as to render a single expression a square. But there are problems where it becomes requisite to find such a value of the unknown term as to render several different expressions squares at the same time. And this is called *double* and *triple equalities.*

Case 1st. As a general equation for double equality, let it be required to find such a value of x that $ax+b$ may be

a square, and the *same value* of x give $cx+d$ a square.

$$\text{Let } ax+b=t^2. \quad \text{Then } x=\frac{t^2-b}{a}.$$

$$\text{And } cx+d=p^2. \quad \text{Then } x=\frac{p^2-d}{c}.$$

$$\text{Therefore } \frac{t^2-b}{a}=\frac{p^2-d}{c}. \quad \text{Or, } ct^2-cb=ap^2-ad.$$

Tranpose cb and multiply by c and we have

$$c^2t^2=acp^2+c^2d-acd.$$

As the left hand side of this equation is a square, *whatever may be the values of c and t*, it is now only requisite to find such a value of p^2 as shall render the other side a square, which can be done by some one of the artifices in the preceding section.

To illustrate this we give the following definite problem.

The double of a certain number increased by 4, makes a square, and five times the same number plus one, also makes a square. What is the number

Let x represent the number.

$$\left.\begin{array}{ll}\text{Then} & 2x+4=t^2 \\ \text{And} & 5x+1=p^2\end{array}\right\} \text{From which} \left\{\begin{array}{l} x=\dfrac{t^2-4}{2} \\ x=\dfrac{p^2-1}{5}\end{array}\right.$$

Hence $5t^2-20=2p^2-2$. Or, $5t^2=2p^2+18$, multiply by 5, and $25t^2=10p^2+90$.

The left hand side of this last equation being a square, *whatever* be the value of t, it is now only necessary to find such a value of p^2 as to make $10p^2+90$ a square, an expression which corresponds to case 7 of the last section. We therefore *cannot proceed unless* we find by trial, by observation, by intuition as it were, some simple value of p that will make $10p^2+90$ a square, and we do perceive that if $p=1$, the expression will become 100, a square.

Now if $p=1$, it will give a definite and positive value to x, and the problem is solved. If not we must find other values of p.

But we have found that $x=\frac{p^3-1}{5}$, and if $p=1$, $x=0$, and the original expressions $2x+4$, and $5x+1$, become 4 and 1, squares it is true, which answer the *technicalities*, but not the spirit of the question.

To find another value of p. Put $p=1+q$. Then $10p^2+90=100+20p+2q^2$. To make this a square assume $100+20q+2q^2=(10-nq)^2=100-20nq+n^2q^2$.

By reduction, $q=\frac{20(n+1)}{n^2-10}$. Now n must be so taken, that n^2 is more than 10: take $n=5$ and $q=8$, $p=9$, then $x=16$ and the original expressions $2x+4=36$, a square, and $5x+1=81$, a square.

CASE 2D. A double equality in the form of $ax^2+bx=\square$ and cx^2+dx, also equals a square, may be resolved by making $x=\frac{1}{y}$, then the expressions will become $\frac{1}{y^2}(a+by)$ and $\frac{1}{y^2}(c+dy,)$ which must be made squares.

But if we multiply a square by a square, or divide a square by a square, the product or quotient will be square.

Now as each of the preceding expressions are to be squares, and as they obviously have a square factor $\frac{1}{y^2}$ it is only necessary to make $a+by$, and $c+dy$ squares, as in the first case.

We may also take another course and assume $ax^2+bx=p^2x^2$, which gives $x=\frac{b}{p^2-a}$, which value put in the other expression, and we have $c\left(\frac{b}{p^2-a}\right)^2+d\left(\frac{b}{p^2-a}\right)=\square$.

Multiplying this by the square $(p^2-a)^2$, and the expression becomes $cb^2-dbd+abp^2=$ some square, from which the value of p can be found and afterwards x.

EXAMPLE.

A certain number added to its square, the sum is a square, and the number subtracted from its square, the remainder is a square. What is the number?

Let $x=$ the number.

Then $x^2+x=\square$. And $x^2-x=$ some other square.

$$\text{Assume } x=\frac{1}{y}. \quad \text{Then } \frac{1}{y^2}\left(1+y\right)=\square.$$

$$\text{And } \frac{1}{y^2}\left(1-y\right)=\square.$$

The problem will be solved if we can find such a value of y, as will at the same time make $1+y$ and $1-y$ squares.

Therefore put $1+y=p^2$, and $1-y=q^2$.

From the first, $y=p^2-1$. And $y=1-q^2$.

Therefore, $q^2=2-p^2$. As q^2 is a square, we have only to find such a value of p^2, as shall render $2-p^2$ a square. But this cannot be done *unless* we can find some simple value of p by inspection, and we do observe it must be one. But p being equal to one, gives $y=0$, which will not answer the conditions. Therefore, let $p=1+t$.

Then $2-p^2=1-2t-t^2=(1-ut)^2=1-2ut+u^2t^2$.

Or, $t=\dfrac{2(u-1)}{u^2+1}$. Take $u=2$. $t=\frac{2}{5}$. $p=1+t=\frac{7}{5}$.

$y=p^2-1=\frac{24}{25}$. $x=\dfrac{1}{y}$. $x=\frac{25}{24}$, a number that will answer the given conditions.

CASE 3D. To resolve a triple equality.

Equations in the form of $ax+by=t^2$, $ax+dy=u^2$, $ex+fy=s^2$, can be resolved thus:

By expunging y, we find $x=\dfrac{dt^2-bu^2}{ad-bc}$.

Then by expunging x, $y=\dfrac{au^2-ct^2}{ad-bc}$.

Substituting these values of x and y in the third equation

and we shall have $\frac{(af-bc)u^2+(de-cf)t^2}{ad-bc}=s^2$.

Assume $u=\pm tz$. Then $u^2=t^2z^2$. Put this value of u^2 in the above, and divide by t^2, and we shall have

$$\frac{(af-bc)z^2+de-cf}{ad-bc}=\frac{s^2}{t^2}.$$

The right hand side of this equation is a square, and therefore all that is now requisite, is to find such a value of z as shall make the other side a square, which when possible, can be done by case 7, section 20.

After z is found t may be assumed of any convenient value whatever. Now u is known, and with t and u known quantities, we know x and y.

The preceding are some of the most comprehensive and general methods yet known; but there are cases in practice where no general rules will be so effectual, as the operator's own judgment and penetration.

Much, *very much* will depend on skill and foresight displayed at the commencement of a problem, by assuming convenient expressions to satisfy one or two conditions at once, and the remaining conditions can be satisfied by some one of the preceding rules.

Examples.

1. It is required to find three numbers in arithmetical progression, such, that the sum of every two of them may be a square.

Let x, $x+y$ and $x+2y$ represent the numbers.

Then by the general formula,

$$2x+y=t^2,\ 2x+2y=u^2,\ 2x+3y=s^2.$$

By exterminating x, we have $\frac{t^2-y}{2}=\frac{u^2-2y}{2}$.

Continuing thus after the general equations, we find a

long and troublesome process, and in conclusion, we find the numbers to be 482, 3362, and 6242.

The above is according to the common as well as the general method.

The following is Mr. Young's Solution.

Let $x—y$, x, and $x+y$ represent the numbers. Then $2x—y$, $2x$, and $2x+y$ must be squares.

Assume $2x=m^2+n^2$, and $y=2mn$.

Then $2x—x=m^2—2mn+n^2$, and $2x+y=m^2+2mn+n^3$ are evidently squares. It therefore only remains to make $2x$ or m^2+n^2 a square, and this can be done as explained at the close of the last section by assuming $m=r^2—s^2$, and $n=2rs$. Then $2x=m^2+n^2=(r^2+s^2)^2$, an expression in which r and s can be assumed in numbers. But they must be *so assumed* that x shall be greater than y to make $x—y$ the first number, positive and for this reason, we must give the literal expressions for the numbers before taking definite values for r and s. The expressions for the numbers are

$$x—y=\tfrac{1}{2}(r^2+s^2)^2—4rs\,(r^2—s^2.)\quad x=\tfrac{1}{2}(r^2+s^2.)^2$$

$$x+y=\tfrac{1}{2}(r^2+s^2)^2+4rs(r^2=s^2.)$$

Take $r=9$, $s=1$, and 482, 3362, 6242, are the numbers.

Another Solution.

Let $\frac{x^2}{2}—y, \frac{x^2}{2}$ and $\frac{x^2}{2}+y$ be the numbers.

Then $x^2—y$, x^2+y, and x^2 must be squares.

But the last being a square, we have only to make $x^2—y$ and x^2+y, squares.

Assume $y=2x—1$. Then $x^2—y=x^2—2x+1$, a square, and we now have only to make $x^2+2x—1$, a square.

Therefore make $x^2+2x—1=(x+n)^2=x^2+2nx+n^2$.

$$x=\frac{n^2+1}{2(1—n)}.$$

It is manifest that n must be less than one, make it .8

Then $x=\frac{1.64}{.4}=\frac{41}{10}$. Or, $\frac{x^2}{2}=\frac{1681}{200}$. $y=\frac{72}{10}=\frac{1440}{200}$.

Then $\frac{482}{400}$, $\frac{3362}{400}$, $\frac{6242}{400}$ are the numbers.

Here we have the numbers expressed in fractions, but the denominators are common, and is a square number, we may therefore multiply all three by 400, and we shall have 482, 3362, and 6242 for the numbers, as in the other solutions.

If we take $n=\frac{5}{6}$ in this last result, we shall have 2162, 7442, and 9442 for the numbers.

2. Find two numbers such, that if to each, as also to their sum, a given square, a^2 be added, the three sums shall all be squares.

Let x^2-a^2 and y^2-a^2 represent the numbers: then the first conditions are satisfied.

It now remains to make $x^2+y^2-2a^2+a^2$ a square, or, $x^2+y^2-a^2=\square$. Assume $y^2-a^2=2ax+a^2$. This assumption will make the expression a square, whatever be the values of either x or a. But the assumed equation gives $y^2=2ax+2a^2$, and as y^2 is a square, we must find *such values* of x and a, as shall make $2ax+2a^2$, a square. Put $x=na$. Then $2na^2+2a^2=\square$, or, $a^2(2n+2)=\square$. Hence it is sufficient that we put $2n+2=$ some square. Therefore, assume $2n+2=16$. Hence $n=7$ and $x=7a$. Now take a equal to any number whatever. If $a=1$, $x=7$, $y=4$, and 48 and 15 are the numbers, add 1 to each, and we have 49 and 16, squares; sum, $63+1=64$, a square.

3. Find three square numbers whose sum shall be a square.

Let $x^2+y^2+z^2=\square$. Assume $y^2=2xz$. Then $x^2+2xz+z^2$ is a square. But $2xz=\square$. Let $x=uz$, then $2uz^2=\square$, or $2u=\square=16$, $u=8$, $x=8z$, $z=1$, $x=8$, $y=4$.

Therefore $64+16+1=81=9^2$.

4. Find three square numbers in arithmetical progression.

Let x^2-y, x^2, and x^2+y represent the numbers. Assume $x^2=y^2+\frac{1}{4}$, then the first and last will be squares, and it only

remains to make the middle term, or $y^2+\frac{1}{4}$, a square. Therefore, put $y^2+\frac{1}{4}=(y-p)^2$, which gives $y=\frac{p^2-\frac{1}{4}}{2p}$. Take $p=1$, then $y=\frac{3}{8}$, and $y^2+\frac{1}{4}=\frac{25}{64}=x^2$. Therefore, $\frac{1}{64}$, $\frac{25}{64}$, $\frac{49}{64}$, are the numbers; but we can multiply them all by the same square number 64, and their *arithmetical* relation will not be changed, and they will still be squares; hence 1, 25, and 49 may be the numbers, or 4, 100, and 196,

5. Find two whole numbers, such that the sum and difference of their squares, when diminished by unity, shall be a square.

Let $x+1=$ one number, and $y=$ the other. Then by the conditions we must make squares of x^2+y^2+2x, and x^2-y^2+2x. Assume $2x=a^2$, and $y^2=2ax$, then the expressions become $x^2+2ax+a^2$, and $x^2-2ax+a^2$, obvious squares, whatever be the values of x and a. But the equations $2x=a^2$ and $y^2=2ax$ must be satisfied. Take $a=4$, then $x=8$, $y=8$, and $x+1=9$. Therefore 9 and 8 are the numbers.

6. Find three whole numbers, such, that if to the square of each, the product of the other two be added, the three sums shall be squares.

Let x, xy, xv, be the numbers. Then by the conditions, x^2+x^2yv, $x^2y^2+x^2v$, $x^2v^2+x^2y$, must be squares. As each term contains a square factor x^2, it will be sufficient to make $1+yv=\square$, $y^2+v=\square$, and $v^2+y=\square$.

Assume $y=4v+4$, and this will make the first and last expressions squares. Substitute this value of y in the second expression, and we shall have $16v^2+33v+16$, which must be made a square. Hence put $16v^2+33v+16=(4-pv)^2$, which reduced gives $v=\frac{33+8p}{p^2-16}$. Take $p=5$, then $v=\frac{73}{9}$. Now take $x=9$, and we have 9, 73, and 328 for the numbers.

7. Find two whole numbers whose sum shall be an inte

gral cube, and the sum of their squares increased by thrice their sum shall be an integral square.

Let $x+y=n^3$, that is some cube. Then $x^2+y^2+3n^3=\square$. Put $2xy=3n^3$, then $x^2+2xy+y^2$ is a square, whatever may be the values of x and y. But x and y must conform to the equations $x+y=n^3$, and $2xy=3n^3$. Work out the value of x from these equations, on the supposition that n is known, and we shall find $2x=n^3+\surd\{n^6-6n^3.\}$

Now x will be rational, provided we can find such a value of n as shall render n^6-6n^3 a square, but if we add 9 to this, we perceive it must be a square, and we have two squares, which differ by 9. Therefore one must be 16, the other 25, as these are the only two *integral squares* which differ by 9. Hence $n^6-6n^3+9=25$. Or, $n^3-3=5$. $n^3=8$, $n=2$, and $x=6$, $y=2$.

8. Find three numbers, such that their sums, and also the sum of every two of them, may all be squares.

Let $x^2-4x=$ the first, $4x=$ second, and $2x+1=$ third. By this notation, all the conditions will be satisfied, except the sum of the last two. That is $6x+1$ must be a square, but to have *three different whole numbers*, no square will answer under 121, the square of 11. Hence put $6x+1=121$. Or, $x=20$. And the numbers will be 320, 80, and 41.

9. Find two numbers, such that their difference may be equal to the difference of their squares, and the sum of their squares shall be a square number.

Let x and y be the numbers. Then $x-y=x^2-y^2$. Divide by $x-y$, and $1=x+y$. Hence $x=1-y$, and $x^2+y^2=1-2y+2y^2$. Which last expression $1-2y+2y^2$ must be made a square. For this purpose, put

$$1-2y+2y^2=(1-ny)^2. \quad \text{Hence } y=\frac{2(n-1)}{n^2-2}.$$

Take n any value to render y less than one in order to

give x a positive value. Therefore take $n=3$, and $y=\frac{4}{7}$ Consequently $x=\frac{3}{7}$, answer.

10. Find three numbers in geometrical progression, such that if the mean be added to each of the extremes, the sums in both cases shall be squares. Ans. 5, 20, and 80.

11. Find three numbers, such, that their product increased by unity shall be a square, also the product of any two increased by unity, shall be a square. Ans. 1, 3, and 8.

Assume 1 for the first number, and x and y for the other

12. Find two numbers, such that if the square of each be added to their product, the sums shall be both squares.

Ans. 9 and 16.

13. Find three integral square numbers in harmonical proportion. Ans. 25, 49, and 1235.

14. Find two numbers in the proportion of 8 to 15, and such that the sum of their squares shall be a square number.

Ans. 136 and 255. Bonnycastle's answer, 476 and 1080.

15. Find two numbers such that if each of them be added to their product, the sums shall be both square.

Ans. $\frac{1}{3}$ and $\frac{5}{3}$.

The above require no explanation from us.

There are many severe and tedious problems in the Diophantine Analysis, proposed by Bonnycastle, Young, and others, which require more time and practice than algebraists in general ought to give for the advantage derived, as time and thought may be better employed in Analytical Geometry, the Calculus, or Astronomy.

THE DIOPHANTINE ANALYSIS.

The Diophantine Analysis is sometimes useful in solving numerical Equations, in which squares and cubes are involved, as the following examples will show.

1. Given $\left\{\begin{matrix} x^2+y=7 \\ y^2-x=7 \end{matrix}\right\}$ to find one value of x and y.

By subtraction, and the transposition of y^2 we have

$$x^2+x+y=y^2 \qquad (a)$$

As the second member of this equation is a square, the first must be a square in *fact*, if not in *form*.

But, we perceive, that if we put $y=x+1$, in the first member, it will be in a square form.

Put this value of y, in the first equation, and we have $x^2+x=6$; which gives $x=2$; hence $y=3$; values which verify both equations.

N. B. This method of operation is not general. It only serves to resolve particular cases. We might have made the first member of equation (a), a square, by putting $y=3x+4$, or $5x+9$; but the results of these substitutions would not verify the primitive equations.

2. Given $\left\{\begin{matrix} 2x^2-3xy+\ y^2=4 \\ x^2+3y^2-2xy=9 \end{matrix}\right\}$ to find values of x and y.

As 4 and 9 are squares, the first members are square in fact, though not in form. But we can make the first members square in form, by assuming

$$2x^2-3xy=0, \quad \text{and } 3y^2-2xy=0.$$

Then $y^2=4$ and $x^2=9$, or $y=2$ and $x=3$; values which verify all the equations.

3. Find such integral values of x, y, and z, as will verify the equations . . . . $x^2+y^2+xy=37$,

and . . $x^2+z^2+xz=49$.

If we add xy to the first equation, and xz to the second, the first members will be square; and, of course, the second members will be square in fact, though not in form.

We have, then, to make $37+xy$, and $49+xz$, squares, to accomplish this.

Put $37+xy=49$, or $xy=12$ . . . . . . (1)

and $49+xz=64$, or $xz=15$ . . . . . . (2)

From (1), . . $x=\frac{12}{y}$; from (2) . . $x=\frac{15}{z}$.

Hence . . $12z=15y$, or, $z=\frac{5y}{4}$.

Take $y=4$, then $z=5$, and $x=3$; values which will verify the given equations.

4. Find such integral values of y and z as shall verify the equation . . . . . $y^2+z^2+yz=61$.

Add yz to both members, then put $61+yz=n^2$.

Now if we assume . . . . $n=8$, $yz=3$.

But $yz=3$ will give $y+z=8$, and these two equations will not give integral values to y and z. Therefore, take $n=9$, then $n^2=81$, $yz=20$, and $y+z=9$. Hence $z=4$ or 5, and $y=5$ or 4.

5. Find such values of x and y as will verify the equations . . . . $xy+xy^2=12$ . . . . (1)

and . . . . $x+xy^3=18$ . . . . (2)

Equation (1) may be put into this form $y^2=\frac{12}{x}-y$ . . (3)

Equation (2) into this . . . . $y^3=\frac{18}{x}-1$ . . . (4)

The first member of equation (4) is a cube; therefore

Put . . . . . . $\frac{18}{x}-1=8$. Whence $x=2$.

6. Given $\left\{\begin{array}{l} v+u+v^2u^2=13 \\ y+u+y^2u^2=21 \\ x+v+x^2v^2=41 \\ x+y+x^2y^2=70 \end{array}\right\}$ to find one value of each of the symbols.

A regular solution would result in a very high and tedious equation; but if the values are *integral*, we can soon determine them as follows:

Take the first equation, and put $v+u=s$, and transpose s. Then $v^2u^2=13-s$; which shows that $13-s$ must be some *square;* and if s is positive, the square cannot be greater than 9. That is, $13-s=9$ or, $s=4$.

Then $v+u=4$, and $vu=3$; giving $v=1$ or 3, and $u=3$ or 1. By taking $u=1$, in the second equation, we find $y=4$. With the values already found we obtain x, from either the third or fourth equations, $v=3$, $u=1$, $x=2$, $y=4$.

7. Given $\left\{\begin{array}{l} 4x^2-2xy=12 \\ 2y^2+3xy=\ 8 \end{array}\right\}$ to find values of x and y.

Put $xy=p$; transpose, &c., we have

$$4x^2=12+2p \quad \text{and} \quad 4y^2=16-6p.$$

Now we must find such a value of p as shall render the second members of these last equations square at the same time; which is $p=2$; this gives $4x^2=16$, $x=2$.

8. Given $\left\{\begin{array}{l} 6x^2+2y^2\ =5xy+12 \\ 3x^2+2xy=3y^2-\ 3 \end{array}\right\}$ to find one value of x and y.

This problem is under (Art. 110) of both editions.

Add the equations together, and reduce, and we have

$$9x^2=y^2+3xy+9.$$

The first member of this equation is a square; therefore the second member is a square, but to make it a square in *form*, as well as in *fact*, we perceive it is only necessary to make

$x=2$. Or call $y^2+3xy+9$ a binomial square, and decide the value of x agreeable to section 8, This gives $x=2$, the answer.

9. Given $\left\{\begin{array}{l} 3x^2-xy=28 \\ 4y^2+3xy=160 \end{array}\right\}$ to find the rational values of x and y. (Y. 139.)

Ans. $x=\pm4$, $y=\pm5$.

There are, and may be equations of the preceding forms which have no rational values, such are not susceptible of of this mode of treatment.

Section XXIII.

Miscellaneous Examples.

1. When wheat was 8 shillings a bushel and rye 5, a man wished to fill his sack for the money he had in his purse. Now if he bought 15 bushels of wheat and laid out the rest of his money in rye, he would want 3 bushels to fill the sack: but if he bought 15 bushels of rye, and then filled his sack with wheat, he would have 15 shillings left. How much of each must he purchase to fill his sack, and lay out all his money?

(Colburn, page 50.) Ans. 10 bushels of each.

Solution by Mr. T. J. Matthews.

Let $x=$ the wheat, and $y=$ the rye. Then it is evident that when he buys 15 bushels of wheat he has too much, as he has not money enough left to fill his sack with rye. Now $15-x$ is the excess of the wheat purchased above what he ought to have had, and this excess of quantity, multiplied by the excess of a bushel of wheat above one of rye, will give the deficiency of his money, or equal to 3 bushels of rye at 5 shillings. Consequently, $3(15-x)=15$, or $x=10$.

By similar reasoning, it will appear that when 15 bushels of rye are purchased, he buys too much, and the excess is $15-y$, which multiplied as before, will equal the excess of

his money, viz: 15 shillings. Therefore $3(15-y)=15$, and $y=10$.

2. A person bought two cubical stacks of hay, for £41; each of which cost as many shillings per solid yard as there were yards in a side of the other, and the greater stood on more ground than the less by 9 square yards. What was the price of each? (Colburn.)

Solution by T. J. Matthews.

Assume $5x$, and $4x$ the sides of the Cubes.

Then $25x^2-16x^2=9$, by the first condition.

Therefore, $x=1$. $125 \cdot 4=500$. $64 \cdot 5=320$.

3 Or, £25. £16. Answer.

3. Given $x+y+z=25$. $xy=6$ $xz=60$, to find x, y and z.

Solution. From the two latter, $z=10y$. Then the first becomes $x+11y=25$. Or $x^2+11xy=25x$, but from the 2d, $11xy=66$. Hence $x^3-25x=-66$. (A)

Assume $2a=-25$, then $6a+9=-66$, and equation (A) becomes $(x^2+2ax+a^2=a^2+6a+9$. Or, $x=3$,)

4. Given $xy=125x+300y$. And $y^2-x^2=90000$, to find x and y. (Young p. 146.) Ans. $x=400$. $y=500$.

Put $y=px$, then $px^2=125x+300px$. (1)

And $p^2y^2-x^2=(300)^2$ (2)

From equation (1) $p=\dfrac{125}{x-300}$. (3)

From equation (2) $p^2-1=\dfrac{(300)^2}{x^2}$

The right hand side of this last equation being a square the other side is also a square, and one accustomed to the analysis will perceive that p must equal $\frac{5}{4}$, to make the expression $p^2-1=\square$. Others can go through the form and they will find that $p=\frac{5}{4}$, which value put in equation (3) gives $x=400$.

It is not imperative that we should resort to the Diaphantine to solve this problem but it is *very convenient.*

5. Find two numbers, such that the fifth power of one may be to the cube of the other, as 972 to 125.

Ans. 6 and 10.

Let x and nx be the numbers.
Then $x^5 : n^3x^3 :: 972 : 125$. Or, $x^2 : n^3 :: 972 : 125$.
Multiply the first and third terms by x, $x^3 : n^3 :: 972x : 125$.

Therefore, $972x = \frac{125x^3}{n^3}$.

The right hand side of this equation is a cube, therefore, $972x=$ a cube: Or, $27 \cdot 36x=$a cube. Hence, $36x$ must be a cube, which it evidently is, when $x=6$, as 36 is the square of 6.

6. Given $\left.\begin{array}{l} x+y+xy(x+y)+x^2y^2=85 \\ xy+(x+y)^2+xy(x+y)=97 \end{array}\right\}$ to find x and y.

Young, 145. Ans. $x=6$ $y=1$.

Put $(x+y)=s$ $xy=p$, then the equations become

$$\left.\begin{array}{l} s+sp+p^2=85 \\ \text{And } p+sp+s^2=97 \end{array}\right\}$$

By addition $(s+p)+s^2+2sp+p^2=182$. (1)

Assume $Q=s+p$, then equation (1) becomes $Q^2+Q=182$. Hence $Q=13$. Or, $s+p=13$. The remaining steps are obvious.

7. Given $\sqrt{x+12} = \frac{12}{\sqrt{x+5}}$, to find x.

Square $x+12 = \frac{144}{x+5}$. Put $x+5=y$.

Then $y+7 = \frac{144}{y}$. Or, $y^2+7y=144$.

Put $2a=7$. Then $18a+81=144$.
Hence $y^2+2ay+a^2=a^2+18a+81$.
$y+a=a+9$. Or, $-a-9$.

8. Given, $\left\{\begin{array}{l}(x+y)(xy+1)=18xy. \\ (x^2+y^2)(x^2y^2+1)=208x^2y^2\end{array}\right\}$ to find x and y.

Ans. $x=2\pm\sqrt{3}$. Or $7\pm4\sqrt{3}$.
$y=7\pm4\sqrt{3}$. Or $2\pm\sqrt{3}$.

Solution. Take $x+y=s$. $xy+1=t$ and $xy=p$.

Then $st=18p$. (1) And $(s^2-2p)(t^2-2p)=208p^2$. (2)

Multiply as indicated and take the value of s^2t^2 from equation (1) and we have $324p^2-2p\,(s^2+t^2)+4p^2=208p^2$.

By reduction, $p=\frac{s^2+t^2}{60}$. From equation (1) $p=\frac{st}{18}$.

Therefore, $\frac{s^2+t^2}{60}=\frac{st}{18}$. Assume $s=nt$.

Then this last equation becomes, by a little reduction,

$\frac{n^2+1}{10}=\frac{n}{3}$. Hence $n=3$. Or, $\frac{1}{3}$.

This establishes a relation between $x+y$ and $xy+1$.

Another Solution by Charles E. Matthews.

Multiply the original equations as indicated, and

$x^2y+xy^2+x+y=18xy$.

$x^4y^2+x^2y^4+x^2+y^2=208x^2y^2$.

Divide the first by xy, the 2d by x^2y^2, then

$x+y+\frac{1}{y}+\frac{1}{x}=18$. And $x^2+y^2+\frac{1}{y^2}+\frac{1}{x^2}=208$.

Assume $x+\frac{1}{x}=m$. And $y+\frac{1}{y}=n$.

Then $m+n=18$. And $m^2+n^2=212$.

From which m and n are easily found, afterwards x and y.

9. A square public green is surrounded by a street of uniform breadth. The side of the square is 3 rods less than 9 times the breadth of the street: and the number of square rods in the street exceeds the number of rods in the perimeter of the square by 228. What is the area of the square?

(Day 307.) Ans. 576 rods.

10. A man wishes to purchase a certain number of acres

of land for the money he has at his command. Cleared land is worth 10 dollars per acre; uncleared land is worth $8.—He finds that if he buys 120 acres of cleared land, and lays out the rest of his money for that which is not cleared, he will not get the quantity of land he wants by 25 acres, but, if he buys 220 acres of uncleared land, and then buys a sufficient number of acres of cleared land to make up the number of acres he wants, he will have 4 dollars left. How many acres of each must he buy to have the quantity he wishes, and lay out all his money? (Harney page 203.

Ans. 20 acres cleared, 218 uncleared.

N. B. Call to mind problem first of this section.

In working geometrical problems algebraically much labor may be saved by paying attention to the *relation* of the given numbers.

We give the following as illustrative of these remarks.

1. If the perimiter of a right angled triangle be 720, and the perpendicular falling from the right angle on the hypothenuse be 144; what are the lengths of the sides?

(Day Alg. p. 305.) Ans. 300, 240 and 180.

If we use the identical numbers given 144 and 720, as nine-tenths of our teachers do, they will give large and tedious equations, but if we compare 144 and 720, we shall perceive that one is exactly 5 times the other, and considering the nature of *similar* triangles, we can work on one of only 144th of the linear dimensions of the first, or a triangle whose perimiter is 5, and perpendicular from the right angle 1.

Solution. Let x and y be the two sides, then $5-x-y$ will be the hypothenuse.

And $x^2+y^2=(5-x-y)^2$, and $xy=5-x-y$.

Each member of this last equation expresses the double area of the triangle. Put $x+y=s$. $xy=p$.

Then $s^2-2p=(5-s)^2=25-10s+s^2$, and $p=5-s$.
Or, $10s\ -2p=25$
And $2s\ +2p=10$

By addit'n $12s\ =35$. Or, $s=\frac{35}{12}$.

But $s=x+y$, the sum of the two sides which, taken from 5, or $\frac{60}{12}$, gives $\frac{25}{12}$, for the hypothenuse of the small triangle, hence $\frac{25}{12}$ by $144=300$, the hypothenuse of the large triangle.

2. The sum of the two sides of a plane triangle is 1155, the perpendicular drawn from the angle included by these sides to the base, is 300; the difference of the segments of the base is 495, what are the length of the three sides?

(Day 305.) Ans. 945, 375, 780.

Write the given members in order, thus 300, 495, 1155. Divide them by 15, and their relation is 20, 33, 77.

The two latter numbers have a common factor 11, which call a. Put $b=20$.

Then the three given lines will be b, $3a$, and $7a$. Let $x=$ the less side. $7a-x=$the greater side, $y=$the shorter segment, and $y+3a=$the longer segment of the base.

Then $y^2+b^2=x^2$. (1

And $y^2+6ay+9a^2+b^2=49a^2-14ax+x^2$. (2)

Subtract (1) from (2), drop $9a^2$ from both sides, and divide by $2a$, and $3y=ab-7x$. (3)

We write $2b$ in place of 40, after dropping $9a^2$.

From the square of (3) subtract 9 times, equation (1,) and we have $-9b^2=a^2b^3-14\ abx+2bx^2$.

Divide by b, afterwards by 2, recollecting that $b=20$, and we have $-90=10a^2-7ax+x^2$.

Add $\frac{9a^2}{4}$, to both sides to complete the square.

Then $\frac{9a}{4}-90=\frac{9}{4}(a^2-40)=\frac{9}{4}\cdot 81=\frac{49a^2}{4}-7ax+x^2$.

Extract square root $\frac{3}{2}\cdot 9=\frac{7a}{2}-x$.

Or, $x=25$. Then $25.15=375$.

3. Divide the number 74 into two such parts that the difference of the square roots of the parts may be 2.

Ans. 25 and 49.

Let $x-1$, and $x+1$ be the square roots, of the two parts. This problem can also be solved by the Diophantine analysis.

4. Given $x^2+y^2=45$ and $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$, to be solved by the Diophantine analysis. Ans. $x=6$. $y=3$.

5. Given $x^2+y^2=45$ and $(x+y)x=54$ to find x and y by the Diophantine analysis. Ans. $x=6$. $y=3$.

The two preceding should also be worked by common algebra.

6. A and B traveled on the same road, and at the same rate, from Huntingdon to London. At the 50th mile stone from London, A overtook a drove of Geese, which were proceeding at the rate of 3 miles in 2 hours, he afterwards met a stage wagon, which was moving at the rate of 9 miles in 4 hours. B overtook the same drove of Geese at the 45th mile stone, and met the same stage wagon exactly forty minutes before he came to the 31st mile stone. Where was B when A reached London?

Solution. Let $x=$ miles traveled by each per hour, and $y=$ distance B was behind A, then $50-2x=$ the distance from London where A met the wagon.

Also, $\frac{y}{x+\frac{9}{4}}=\frac{4y}{4x+9}=$ time elapsed between the meeting of the wagon with A and B, therefore $\frac{9y}{4x+9}=$ distance traveled by the wagon during this time, consequently

$50-2x+\frac{9y}{4x+9}=$distance of the wagon from London, when

met by B. But this distance is also $=31+\frac{2x}{3}$ therefore

$50-2x+\frac{9y}{4x+9}=31+\frac{2x}{3}$. Again $y+5:5::x:\frac{3}{2}$,

whence $3y+15=10x$ and $y=\frac{10x-15}{3}$, substituting and reducing, we get the quadratic

$x^2-\frac{123x}{16}=\frac{378}{32}$. Whence $x=9$, consequently $y=25$.

7. Given $x^2+xy=77$, and $xy-y^2=12$, to find x and y, by the Diophantine analysis. Ans. $x=7$. $y=4$.

8. Three equal circles touch each other externally, and enclose between the points of contact a acres of ground, what are the radii of the circles?

Ans. $\left(\frac{a}{0{,}16125}\right)^{\frac{1}{2}}$

9. A person has £27 6*s.* in guineas and crown pieces, out of which he pays a debt of £14 17*s.*, and finds that he has exactly as many guineas left as he has paid crowns away; and as many crowns as he has paid away guineas; how many of each had he at first?

Ans. 9 crowns paid away; 12 guineas paid away.

Suppose $x=$ the guineas paid away.

And $y=$ the crowns paid away.

$$\left.\begin{array}{l}\text{Then } 21x+5y=297=\text{ amount paid out}\\ \text{And } 5x+21y=249=\text{ amount on hand}\end{array}\right\}\text{ per question.}$$

Add these equations and divide by 26, &c.

10. The sum of three numbers in harmonical proportion is 191, and the product of the first and third is 4032. What are the numbers? Ans. 72, 63, 56.

11. Is it possible to pay £50 by means of guineas and three shilling pieces only. Ans. Impossible.

12. A merchant drew every year upon the stock he had in trade, the sum of a dollars for the expense of his family.

His profits each year, were the nth part of what remain-

ed after this reduction, but at the end of the 3d year he finds his stock exhausted; how much had he at the beginning;

Ans. $\frac{a(3n^2+3n+1)}{(n+1)^2}$.

10. Given $\left\{\begin{array}{r}\sqrt{(5\sqrt{x}+5\sqrt{y})}+\sqrt{x}+\sqrt{y}=10 \\ \text{And } x^{\frac{5}{2}}+y^{\frac{5}{2}}=275\end{array}\right\}$ to find x and y.

Put $\sqrt{(5\sqrt{x}+5\sqrt{y})}=n$. Then the first equation becomes $\frac{n^2}{5}+n=10$. Which equation gives $n=5$.

Whence $\sqrt{x}+\sqrt{y}=5$.

From this last and the 2d equation, we find $x=9$, and $y=4$.

The following are from Bland's Problems, and involve equations only of the second degree. They are too severe for learners, but we are tempted to leave one or two of them, without solution, for the benefit of those who deem keys unnecessary. More of like character might be given.

$$x^{\frac{1}{2}}-\frac{8}{x}=\frac{7}{\sqrt{x}-2}. \qquad \text{Ans. } x=16 \text{ or } 1.$$

$$2x^{\frac{3}{2}}(x^3+a^3)^{\frac{1}{2}}=2x^2(x+2a)+a^2(x-a).$$

Ans. $x=\frac{1}{2}a$ or $-a$.

$$x+(3y^2-11+2x)^{\frac{1}{2}}=7+2y-y^2;$$

$$(3y-x+7)^{\frac{1}{2}}=\frac{x+y}{x-y}. \qquad \text{Ans. } x=4,\ y=2.$$

Double the first equation, transpose $-2y^2$, and subtract 11 from both members, then we have

$$2y^2+2x-11+2(3y^2-11+2x)^{\frac{1}{2}}=3+4y.$$

Add y^2 to both members, and conceive the terms in the vinculum to be P; then

$$P^2+2P=3+4y+y^2;$$

$$\text{Or,} \qquad P+1=2+y.$$

By restoring the value of P, and reducing, we have

$x=6+y-y^2$. Put this in the 2d eq., &c.

4. Given $x^2y-4=4x^{\frac{1}{2}}y-\frac{1}{4}y^3$,

and $x^{\frac{3}{2}}-3=x^{\frac{1}{2}}y^{\frac{1}{2}}\left(x^{\frac{1}{2}}-y^{\frac{1}{2}}\right)$, to find x and y.

Ans. $x=1$, $y=4$.

Put $x^{\frac{1}{2}}=P$, $y^{\frac{1}{2}}=Q$, and we have

$$P^4Q^2-4=4PQ^2-\tfrac{1}{4}Q^6, \quad \ldots\ldots\ldots (1)$$

$$P^3-3 \quad =PQ(P-Q), \quad \ldots\ldots\ldots (2)$$

Now put $P=nQ$, and eq. (1) becomes

$$(4n^4+1)Q^6-16nQ^3=16.$$

Conceive n to be a known quantity, then the last equation is quadratic, and a solution gives

$$Q^3=\frac{4(2n^2+2n+1)}{4n^4+1}=\frac{4}{2n^2-2n+1}=\frac{4}{2n^2-2n+2-1}.$$

But from (2), $Q^3=\dfrac{3}{n^3-n^2+n}=\dfrac{3}{n(n^2-n+1)}$.

Put the two values of Q^3 equal, and put $n^2-n+1=R$, (3)

Then $\dfrac{4}{2R-1}=\dfrac{3}{nR}$. Whence $2n=\dfrac{6R-3}{2R} \quad \ldots\ldots (4)$

But from (3), resolved as a quadratic,

$$2n=1\pm\sqrt{4R-3} \quad \ldots\ldots\ldots\ldots (5)$$

From (4) and (5), $2R\pm2R\sqrt{4R-3}=6R-3$;

Or $\ldots\ldots$ $\pm2R\sqrt{4R-3}=4R-3$.

Put $\ldots\ldots$ $\sqrt{4R-3}=S$,

Then $S^2\pm2RS=0$, Or, $S(S\pm2R)=0$.

This last equation may be verified by taking either factor equal to zero; and as the first factor only gives a rational quantity, we take that which gives $R=\frac{3}{4}$.

By retracing, we easily find x and y.